Higher logarithmic topological quantum field theories and the residue torsion of manifolds

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Higher Logarithmic Topological Quantum Field Theories and the Residue Torsion of Manifolds

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"The pine fought the storm and broke.
The willow yielded to the wind and snow and did not break."

Kanō Jigorō Shihan

Abstract

Starting from Scott's recent work on Logarithmic Topological Quantum Field Theories (*LogTQFTs*, [**72**]), we will show that the Euler characteristic of a manifold with boundary is another instance of a topological invariant arising as a character of a LogTQFT. Along the way, we will prove a classification theorem for 2-dimensional LogTQFTs and study the additivity (with respect to gluing) of the index of Dirac operators from the point of view of the boundary integrals.

In Part II, we will generalize the ideas and concepts in Part I and introduce *Higher LogTQFTs*, which will be defined as log-functors on subcategories of $\mathbf{Cob}_n$, the category of n-dimensional cobordisms. Such log-functors take values in the cyclic homology of a representation of $\mathbf{Cob}_n$ and will be, in most cases, obtained by composition with Chern characters. This generalization appears natural in the light of the functorial construction of a LogTQFT and provides a tool to capture finer additive invariants of manifolds which arise from the presence of additional data, such as a fibering of the manifold or a group action on a covering. The family and Novikov signatures will be shown to be two key examples of characters of higher logTQFTs and their additive nature will arise as a consequence of this.

Finally, in Part III, we will define a new log-structure called residue analytic torsion, in analogy with Ray-Singer analytic torsion, and introduced for the first time by Scott in his last work, [**72**]. It is defined via Wodzicki residue trace, hence the name. We will show a classification theorem for residue torsion on manifolds (with and without boundary) and relate this results to Index Theory and LogTQFTs. Moreover, it will also be possible to extend such torsion to fibre bundles and characterize it in terms of Higher LogTQFTs, in the spirit of Part II.

Contents

Introduction

The study presented in this thesis originated from the construction and investigation of a new algebraic theory, or categorification, of logarithmic representations and their log-determinant characters contained in [**72**], after the analysis and observations of [**73**]. This is our starting point: [**72**] and its key definition of *log-functors*, i.e. simplicial maps between (suitably defined) simplicial sets with a log-additive property. There, some fundamental examples are investigated, among which the representation of the topological signature as the log-determinant of a $LogTQFT$. Our goal here is to add new examples and extend the general theory.

The purpose of defining such categorical logarithms is mainly to capture those manifold invariants that behave additively with respect to gluing of manifolds along a common boundary component. As such, these invariants can be seen as semi-classical, as they can be located between classical bordism invariants (*genera*)

$$\mu : \Omega_* \to R,$$

i.e. ring homomorphisms on the Thom ring Ω_* of bordism classes of closed manifolds, and quantum bordism invariants ($TQFTs$)

$$Z : \mathbf{Cob}_n \to \mathbf{Vect}_{\mathbb{F}},$$

i.e. symmetric monoidal functors from the category of n-dimensional cobordisms $\mathbf{Cob}_n$ to the category of vector spaces.

In this introduction, we present the structure of our exposition and report the main results.

Chapter 0:

We start with an introductory chapter, whose main purpose is to set the notation and recall some standard results. There and for the rest of the exposition, we will consider compact C^∞-manifolds X (which will be simply called *manifolds*), possibly with non-empty smooth boundary $Y := \partial X$ and a collar neighbourhood

$U := [0,1) \times Y$ with *product structure*

$$g^X_{|U} = dt^2 + g^Y, \qquad g^X \text{ Riemannian metric for } X,$$

where coordinates $x = (t,y)$ are chosen in such a way that $y \in Y$ and $t \in [0,1)$ corresponds to an inward-pointing normal direction. For $E \to X$ a Hermitian vector bundle, $C^\infty(X,E)$ will denote the space of smooth sections and $H^s(X,E)$ the associated Sobolev space. Sections $s \in C^\infty(U, E_{|U})$ restrict to the boundary via a continuous operator

$$\gamma : H^s(X,E) \to H^{s-\frac{1}{2}}(Y,E'), \qquad (\gamma s)(y) := s(0,y), \qquad E' := E_{|Y},$$

which will be used to study boundary value problems. $\Psi^{\mathbb{Z}}(X,E) := \bigcup_{m\in\mathbb{Z}} \Psi^m(X,E)$ will denote the algebra of integer order *classical pseudodifferential operators* (*classical ψdos*), and $\Psi^{-\infty}(X,E) := \bigcap_{m\in\mathbb{R}} \Psi^m(X,E)$ the ideal of *smoothing* ψdos, i.e. those A whose Schwartz kernel $k^A(x,y)$ is smooth. We recall in this context that $\Psi^{-\infty}(X,E)$ has a (projectively) unique trace, called *classical*, defined by the integral $\mathrm{Tr}(A) := \int_X \mathsf{tr}\ k^A(x,x)dx$.

We will mainly consider the bundle of differential forms $\Lambda(X) \to X$, with sections $\Omega(X) := C^\infty(X, \Lambda(X))$. When $Y \neq \emptyset$, the restriction $\gamma\omega \in C^\infty(Y, \Lambda^k(X)_{|Y})$ decomposes as

$$\gamma\omega = \omega_1 + dt \wedge \omega_2, \qquad \omega_1 \in \Omega^k(Y), \quad \omega_2 \in \Omega^{k-1}(Y),$$

and defines the two orthogonal projections $\mathcal{R}\gamma\omega = \omega_1$ and $\mathcal{A}\gamma\omega = \omega_2$. They commute with the exterior derivative d and codifferential δ, respectively, and refine the complexes $(\Omega^k(X), d)$ and $(\Omega^k(X), \delta)$ to

$$d_k : \Omega^k_{\mathcal{R}}(X) \to \Omega^{k+1}_{\mathcal{R}}(X) \qquad \text{and} \qquad \delta_k : \Omega^{k+1}_{\mathcal{A}}(X) \to \Omega^k_{\mathcal{A}}(X),$$

where $\Omega^k_{\mathcal{R}}(X) = \{\omega \in \Omega^k(X) |\ \mathcal{R}\gamma\omega = 0\}$ and $\Omega^k_{\mathcal{A}}(X) = \{\omega \in \Omega^k(X) |\ \mathcal{A}\gamma\omega = 0\}$, i.e. $\Omega^k(X)$ with *relative*, resp. *absolute*, boundary conditions. Let $H^k_{\mathcal{R}}(X,\mathbb{C})$ and $H^k_{\mathcal{A}}(X,\mathbb{C})$ be the cohomology of $(\Omega^k_{\mathcal{R}}(X), d)$ and $(\Omega^k_{\mathcal{A}}(X), \delta)$, respectively. Then, by de Rham theorem (§4.1, [**23**]),

$$\mathcal{H}^k(X,Y) \cong H^k_{\mathcal{R}}(X,\mathbb{C}) \qquad \text{and} \qquad \mathcal{H}^k(X) \cong H^k_{\mathcal{A}}(X,\mathbb{C}),$$

and the Euler characteristic and relative Euler characteristic can be respectively defined as

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim\ H^k_{\mathcal{A}}(X,\mathbb{C}) \quad \text{and} \quad \chi(X,Y) = \sum_{k=0}^{n} (-1)^k \dim H^k_{\mathcal{R}}(X,\mathbb{C}).$$

The end of the chapter recalls the relationship $\chi(X) = \chi(X,Y) + \chi(Y)$ and the quasi-additivity of the Euler characteristic:

$$\chi(X_1 \cup_N X_2) = \chi(X_1) + \chi(X_2) - \chi(N), \qquad N := \partial X_1 \cap \partial X_2,$$

which is proper additivity when X_1, X_2 have *even* dimension, as in this case $\chi(N) = 0$.

Chapter 1:

Having set the basic analytical and topological notations, in Chapter 1 we establish definitions and properties of log, trace, and det. Thus, a *logarithmic representation* of a semigroup $\mathcal{S}$ in an algebra $\mathcal{T}$ is a homomorphism $\log : \mathcal{S} \to \mathcal{T}/[\mathcal{T},\mathcal{T}]$ with a log-additive property $\log ab = \log a + \log b$; a *trace* is a homomorphism of groups $\tau : (\mathcal{T},+) \to (\mathcal{U},+)$ such that $[\mathcal{T},\mathcal{T}] \subset \ker(\tau)$; and a *determinant* is the composition $\det_{\tau,e} := e \circ \tau \circ \log$, where $e : (\mathcal{U},+,\cdot) \to (\mathcal{V},+,\cdot)$ is an exponential map, i.e. a homomorphism of unital rings such that $e(a+b) = e(a) \cdot e(b)$. In particular, the term *log-determinant* (or *log-character*) will stand for a composition $\tau \circ \log : \mathcal{S} \to \mathcal{U}$. In this generic context, we prove some equivalent criteria for the uniqueness of trace, log, and det (Lemmas 1.2.4, 1.2.5, and 1.2.6). Then we recall two known examples of log-structures: the *global* logarithm on $\mathrm{GL}(n,\mathbb{C})$ and the index of a Fredholm operator $A \in \mathrm{Fred}(H)$ on a Hilbert space H. The latter can, in fact, be obtained from a logarithm $\log : \mathrm{Fred}(H) \to \mathcal{F}(H)$ defined as $\log A := [A,P]$, for P any parametrix and $\mathcal{F}(H)$ is the ideal of finite rank operators, by composition with the classical trace Tr on $\mathcal{F}(H)$.

The rest of Chapter 1 is then devoted to the presentation of log-functors and is a summery of the core of **[72]**. In order to define log-functors, the starting point is a *monoidal product representation* (Definition 1.4.13) of a symmetric monoidal category $(\mathbf{C},\otimes)$, which is defined to be a functor $F : \mathbf{C} \to \mathbf{B}$, $\mathbf{B}$ an additive category, together with *insertion transformations*, i.e. morphisms

$$\eta_{\otimes y} c : F(c) \to F(c \otimes y), \qquad c, y \in \mathrm{obj}(\mathbf{C})$$

that are compatible with $\otimes$, i.e. $\eta_{\otimes(y\otimes y')} c = \eta_{\otimes y'}(c \otimes y) \circ \eta_{\otimes y} c$, and are compatible with commutation, i.e. $\eta_{\otimes(y\otimes y')} c = \mu_\sigma(c \otimes y' \otimes y) \circ \eta_{\otimes(y\otimes y')} c$. Here, $\mu_\sigma(x)$ is a canonical isomorphism $F(x) \to F(x_\sigma)$, where $x_\sigma := x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, i.e. the object $x := x_1 \otimes \cdots \otimes x_n \in \mathrm{obj}(\mathbf{C})$ after the action of a permutation $\sigma \in S_n$. Insertion morphisms $\eta_{\otimes y}$ intertwine with permutation isomorphisms $\mu_\sigma(x)$, thus combining into more elaborated insertion morphisms, denoted η_y^k, which are used to define a *presimplicial structure* on the image $F(\mathbf{C})$ (which is the reason why we need monoidal product representations). Specifically, if $\mathrm{obj}(\mathbf{C}^p)$ denote the set of p-tuples $x_0 \otimes \cdots \otimes x_{p-1}$ of objects of $\mathbf{C}$, then such presimplicial structure is defined by p-simplices

$$\Delta_p = \{(\xi, x_0, \ldots, x_{p-1}) \mid \xi \in F(x_0 \otimes \cdots \otimes x_{p-1}), x_j \in \mathrm{obj}(\mathbf{C})\} \subset \mathrm{obj}(\mathbf{B}) \times \mathrm{obj}(\mathbf{C}^p).$$

degeneracy maps $s_k(w) : \Delta_p \to \Delta_{p+1}$

$$s_k(w)(\xi, x_0, \ldots, x_{p-1}) := (\eta^k_w(\xi), x_0, \ldots, x_{k-1}, w, x_k, \ldots, x_{p-1})$$

(it is presimplicial exactly because face maps may not be available).

Once a monoidal product representation is chosen, we can define the key object of our study (Definition 1.4.28):

Definition. Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category and $F : \mathbf{C} \to \mathbf{Ring}$ a monoidal product representation. Then a *log-functor* is a presimplicial map

$$\log : \mathcal{N}\mathbf{C} \to F(\mathbf{C})/[F(\mathbf{C}), F(\mathbf{C})],$$

$$\log_{x\otimes y} : \mathrm{mor}(x, y) \to \frac{F(x \otimes y)}{[F(x \otimes y), F(x \otimes y)]}, \quad \alpha \mapsto \log_{x\otimes y} \alpha, \quad x, y \in \mathrm{obj}(\mathbf{C})$$

such that if $\alpha \in \mathrm{mor}(x, y)$ and $\beta \in \mathrm{mor}(y, z)$, then

$$\widetilde{\eta}_y(\log_{x\otimes z} \beta \circ \alpha) = \widetilde{\eta}_{\otimes z}(\log_{x\otimes y} \alpha) + \widetilde{\eta}_{x\otimes}(\log_{y\otimes z} \beta)$$

in $F(x \otimes y \otimes z)/[F(x \otimes y \otimes z), F(x \otimes y \otimes z)]$.

It is then clear why we need insertion maps: each logarithm lives in a different space, hence $\log_{x\otimes y} \alpha$ and $\log_{y\otimes z} \beta$ can be added together only if represented into a common space $F(x\otimes y\otimes z)$. The object $\mathcal{N}\mathbf{C}$ is the *nerve* of $\mathbf{C}$, a presimplicial set naturally obtained from $\mathbf{C}$ as follows: the space of p-simplices $\mathcal{N}_p\mathbf{C}$ is composed by p-tuples of morphisms $(\alpha_0, \ldots, \alpha_{p-1})$, $\alpha_j \in \mathrm{mor}(x_j, x_{j+1})$ with $j \in \{0, \ldots, p-1\}$, and the degeneracy maps $s_j : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p+1}\mathbf{C}$ are defined as:

$$s_j(\alpha_0, \ldots, \alpha_{j-1}, \alpha_j, \ldots, \alpha_{p-1}) := (\alpha_0, \ldots, \alpha_{j-1}, \mathrm{id}_{x_j}, \alpha_j, \ldots, \alpha_{p-1}),$$

$\mathrm{id}_{x_j} : x_j \to x_j$ the identity. $F(\mathbf{C})/[F(\mathbf{C}), F(\mathbf{C})]$ is an an abelian category induced by the canonical projection $\pi : R \to R/[R, R]$, R a ring, and inherits a presimplicial structure from $F(\mathbf{C})$. In particular, when $\mathbf{C} = \mathbf{Cob}_n$ a log-functor is called *a Logarithmic Topological Quantum Field Theory* (LogTQFT for short) of dimension n, in analogy with a TQFT. It is important, though, to remark that a LogTQFT is not a symmetric monoidal functor, but a functor ∞-categories. However, it can be used to define one, at least in a weak sense:

LEMMA. Let $F : \mathbf{Cob}_n \to \mathbf{Ring}$ be an unoriented monoidal product representation (i.e. $F(M) = F(M^-)$, where M^- means that the opposite orientation is considered) with trace $\tau_c : \mathrm{end}_{\mathbf{A}}(a_c) \to \mathrm{end}_{\mathbf{A}}(1)$, defining a LogTQFT $\log : \mathcal{N}\mathbf{Cob}_n \to F(\mathbf{Cob}_n)/[F(\mathbf{Cob}_n), F(\mathbf{Cob}_n)]$. If $\epsilon : \mathrm{end}_{\mathbf{A}}(1) \to \mathbb{F}$ is an exponential map into a field, then there exists a scalar-valued TQFT $Z_{\log,\tau,\epsilon}$ defined as $Z_{\log,\tau,\epsilon}(M) = \mathbb{F}$ for $M \in \mathrm{obj}(\mathbf{Cob}_n)$ and $Z_{\log,\tau,\epsilon}(\overline{W}) = \epsilon(\tau(\log \overline{W}))$ for $\overline{W} \in \mathrm{mor}(\mathbf{Cob}_n)$.

As stated in the Lemma, the conversion from a LogTQFT to a TQFT requires a *categorical trace* τ, i.e. there exist $c \in \mathrm{obj}(\mathbf{C})$ for which we have a ring homomorphism $\tau_c : F(c) \to \mathrm{end}(1)$ such that the trace property holds: if $\alpha \in \mathrm{mor}(F(c), F(c'))$, $\beta \in \mathrm{mor}(F(c'), F(c))$ such that $\beta \circ \alpha \in \mathrm{end}(F(c))$ and $\alpha \circ \beta \in \mathrm{end}(F(c'))$, then $\tau_c(\beta \circ \alpha) = \tau_{c'}(\alpha \circ \beta)$. We will need to require traces to be F-compatible, i.e. $\forall c \in \mathrm{obj}(\mathbf{C})$, τ_c satisfies $\tau_{c\otimes y} \circ \eta_{\otimes y} c = \tau_c$ and $\tau_{x_\sigma} \circ \mu_\sigma(x) = \tau_x$. It will follow that τ_c factors though $\pi_c : F(c) \to F(c)/[F(c), F(c)]$, i.e. $\tau_c = \widetilde{\tau}_c \circ \pi_c$. Moreover, the trace $\widetilde{\tau}$ on $F(\mathbf{C})/[F(\mathbf{C}), F(\mathbf{C})]$ satisfies the analogous compatibility condition $\widetilde{\tau}_{c\otimes y} \circ \widetilde{\eta}_{\otimes y} c = \widetilde{\tau}_c$.

As we will see clearly in the next Chapter 2, traces will yield manifold invariants as log-determinants of LogTQFTs. In fact, the τ-character of a LogTQFT defines a *log-determinant functor representation* of $\mathbf{Cob}_n$, i.e. $\widetilde{\tau}_{M\sqcup M'} \circ \log_{M\sqcup M'} \overline{W}$, which will be independent of insertion maps:

$$\widetilde{\tau}_{M\sqcup M'}(\log_{M\sqcup M'} \overline{W}) = \widetilde{\tau}_{M\sqcup M'\sqcup M}(\eta_{M''} \log_{M\sqcup M'} \overline{W}).$$

Additivity follows from log-additivity:

$$\widetilde{\tau}(\log \beta\alpha) = \widetilde{\tau}(\log \alpha) + \widetilde{\tau}(\log \beta), \qquad \alpha \in \mathrm{mor}(c, c'), \beta \in \mathrm{mor}(c', c'').$$

Finally, we conclude the chapter with a classification result we were able to prove for LogTQFT of dimension 2, which we called *Unoriented Logarithm Theorem for Orientable Surfaces* (Corollary 1.4.42). Here, unoriented LogTQFT means $\log_{M\sqcup M'} \overline{W} = \log_{M\sqcup M'} \overline{W^-}$, i.e. the logarithm is invariant under change of orientation. The theorem shows that a 2-dimensional LogTQFT is fully characterized by its definition on the disc:

THEOREM. Let $F : \mathbf{Cob}_2 \to \mathbf{Ring}$ be an unoriented monoidal product representation and let $\log : \mathcal{N}\mathbf{Cob}_2 \to F(\mathbf{Cob}_2)/[F(\mathbf{Cob}_2), F(\mathbf{Cob}_2)]$ be an unoriented LogTQFT. Let $\Sigma_{g,k}$ denote an orientable, compact, and connected surface of genus

g, whose boundary $\partial\Sigma_{g,k}$ has k connected components, i.e. $\partial\Sigma_{g,k} \cong \bigsqcup_k S^1$. Then, $\forall g, k \in \mathbb{N}$:

$$\log_{\bigsqcup_k S^1} \overline{\Sigma}_{g,k} = \chi(\Sigma_{g,k}) \cdot \widetilde{\eta}_{\bigsqcup_{j=1}^{k-1} S^1} \log_{S^1} \overline{D},$$

where $\chi(\Sigma_{g,k}) = \chi(\Sigma_g) - k$ is the Euler characteristic of $\Sigma_{g,k}$ and $\chi(\Sigma_g)$ is the closed surface Σ_g obtained from $\Sigma_{g,k}$ by gluing k discs along the boundary components.

Chapter 2:

In this chapter, we prove that the Euler characteristic of an even dimensional manifold can be viewed as a log-determinant of a LogTQFT. Since the result is based on the index of a Dirac operator with boundary conditions, we started the section by recalling the main ingredients of Elliptic Boundary Value Problems (EBVPs), which we briefly summarize here. The Dirac operator we will be working with is the *de Rham* operator

$$\eth := (d+\delta)^+ : \Omega^+(X) \to \Omega^-(X)$$

on X, considered with non-empty boundary Y, and relative to a $\mathbb{Z}_2$-grading of $\Omega(X)$ into even and odd order smooth forms.

The crucial observation is that, if $Y = \emptyset$, then $(d+\delta)^+$ is Fredholm and $\operatorname{ind}(d+\delta)^+ = \chi(X)$. When $Y \neq \emptyset$, then a similar result holds but we need suitable boundary conditions. A natural class of boundary conditions is represented by the APS (ψdodifferential) projections, $\Pi_{\lambda \geq a} : L^2(Y, \Lambda(X)_{|Y}) \to V_{\geq a} := \bigoplus_{\lambda \geq a} V_\lambda$, $a \in \mathbb{R}$. Here, V_λ is the λ-eigenspace of an elliptic self-adjoint operator $\mathcal{B}$ on Y which originates from the decomposition of $\eth$ into $\sigma(\partial_t + \mathcal{B})$ on a neighbourhood U of Y. $\sigma := c(dt)$ is the Clifford multiplication associated to the normal coordinate t and, by assuming a product structure, $\mathcal{B}$ is independent of t and corresponds to the restriction of $\eth$ to Y.

APS projections are not the main focus here (they are in [**72**], where they are needed to show that the topological signature of X is the trace-character of a LogTQFT), but they are close to a key ingredient for EBVPs: the Calderón projection $\mathcal{C} \in \Psi^0(Y, \Lambda^+(X)_{|Y})$, which is defined in the following way. Our bundle $\Lambda(X)$ and Dirac operator $\eth$ are assumed to be the restriction to X of a bundle $\Lambda(\widetilde{X})$ and Dirac operator $\widetilde{\eth}$ over a *closed* (i.e. without boundary) manifold $\widetilde{X}$, into which X embeds, such that $\widetilde{\eth}$ is invertible (this can always be obtained by taking the 'double' of X). Then the Calderón projection $\mathcal{C}$ is defined as the operator $\gamma r \widetilde{\eth}^{-1} \widetilde{\gamma}^* \sigma$, where r is the restriction to X and $\widetilde{\gamma}$ is the restriction operator from $\widetilde{X}$ to Y. It is the projection onto the space of *Cauchy data*, i.e. the restriction to Y of $\ker \eth$. All boundary conditions that are *well-posed* in the sense of EBVPs (Definition 2.2.2) will be of the form $\mathcal{P}\gamma\omega = 0$, for $\mathcal{P}$ a smooth perturbation of $\mathcal{C}$, i.e. a projection $\mathcal{P} \in \Psi^0(Y, \Lambda^+(X)_{|Y})$ such that $\mathcal{P} = \mathcal{C} + \mathcal{S}$, with $\mathcal{S} \in \Psi^{-\infty}(Y, \Lambda^+(X)_{|Y})$. For instance, $\mathcal{C} - \Pi_{\geq a} \in \Psi^{-\infty}(Y, \Lambda^+(X)_{|Y})$ when a product structure is assumed.

Our interest in EBVPs lies in the fact that for such well-posed boundary conditions the realization $\eth_{\mathcal{P}}$, i.e. the unbounded operator acting as $\eth$ on the space $\{\omega \in H^1\Omega^+(X) | \mathcal{P}\gamma\omega = 0\}$, is Fredholm and its index equals the one of another Fredholm operator: the Toeplitz-type operator $\mathcal{PC} : \mathrm{ran}\mathcal{C} \overset{\mathcal{P}}{\to} \mathrm{ran}\mathcal{P}$. Hence, $\mathrm{ind}(\eth_{\mathcal{P}}) = \mathrm{ind}(\mathcal{PC})$ and the information is concentrated on the boundary. It is this boundary dependence that we look for when searching for a LogTQFT.

Another fundamental property of the index is quasi-additivity, another feature that could arise from a LogTQFT. In fact, if two n-dimensional manifolds X_i with boundary $Y_{i-1} \sqcup Y_i$, $i = 1, 2$ are glued along the common boundary Y_1 into the manifold $X = X_1 \cup_{Y_1} X_2$, then:

$$(\star) \qquad \mathrm{ind}\ \eth_{\mathcal{P}} = \mathrm{ind}\ \eth_{1\mathcal{P}_1} + \mathrm{ind}\ \eth_{2\mathcal{P}_2} + \mathrm{ind}\ \mathcal{Q}$$

where $\mathcal{Q}$ is a Fredholm operator on the boundary component Y_1, and $\eth_{\mathcal{P}}, \eth_{i\mathcal{P}_i}$ are realization of the restrictions of $\eth$ to X and X_i with respect to well-posed boundary conditions. Since formula $(\star)$ is usually found in the literature for X a closed manifold, we proved it but from the point of view of Calderón projections (Theorem 2.4.15). To our knowledge, such an approach had not been previously investigated. Formula $(\star)$ becomes a proper additivity in some cases, for example when we consider only *relative* boundary conditions (the operator $\mathcal{R}$ described in Chapter 0 that selects the tangential component of the decomposition of a smooth form to the boundary). In this case, $\mathrm{ind}\mathcal{Q} = \chi(Y_1)$, which vanishes if n is even. Finally, $\mathrm{ind}(\mathcal{PC})$ can be seen as a trace-character. In fact, well-posedness yields $\mathrm{ind}(\mathcal{PC}) = \mathrm{Tr}(\mathcal{C} - \mathcal{P})$ (lemma 3.8, [**72**]).

Thus, we define an even dimensional LogTQFT as follows. First, we consider the representation $F_{-\infty} : \mathbf{Cob}_{2n} \to \mathbb{C}\text{-}\mathbf{Alg}$ defined as $F_{-\infty}(M) := \Psi^{-\infty}(M, \Lambda^+(M))$, which is unoriented and has a trace $\mathrm{Tr}_M : F_{-\infty}(M) \to \mathbb{C}$. Then, we define a log-functor $\log^\chi : \mathcal{N}\mathbf{Cob}_{2n} \to F_{-\infty}(\mathbf{Cob}_{2n})/[F_{-\infty}(\mathbf{Cob}_{2n}), F_{-\infty}(\mathbf{Cob}_{2n})]$ by setting

$$\log^\chi_{M_0 \sqcup M_1}(\overline{W}) := \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp(\mathcal{C}_W - \mathcal{R}_{\partial W}) \in \frac{F_{-\infty}(M_0 \sqcup M_1)}{[F_{-\infty}(M_0 \sqcup M_1), F_{-\infty}(M_0 \sqcup M_1)]},$$

with $\overline{W} \in \mathrm{mor}_{\mathbf{Cob}_{2m}}(M_0, M_1)$, $\kappa_\sharp : F_{-\infty}(\partial W) \to F_{-\infty}(M_0 \sqcup M_1)$ a natural isomorphism, and $\pi_{M_0 \sqcup M_1}$ the projection to the quotient. Its Tr-character is

$$\mathrm{Tr}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W})\right) = \mathrm{Tr}\left(\mathcal{C}_W - \mathcal{R}_{\partial W}\right)) = \mathrm{ind}\mathfrak{d}_{\mathcal{R}} = \chi(X, Y)$$

and additivity follows from the additivity of the relative Euler characteristic in even dimension (Theorem 2.5.5). In conclusion, we remark the consistency of this result with the Unoriented Logarithm Theorem for Orientable Surfaces. This concludes Chapter 1 and the first part of the thesis.

Chapter 3:

This is the first chapter of the second part of the thesis, where we extend by functoriality the definition and properties of LogTQFTs and their trace-characters. In fact, the category $F(\mathbf{C})/[F(\mathbf{C}), F(\mathbf{C})]$ has a presimplicial structure inherited from $F(\mathbf{C})$ by composition with the covariant functor Π induced by the projection onto the quotient $\pi_c : F(c) \to F(c)/[F(c), F(c)]$; the latter is where a logarithm $\log_c \alpha$ lives. Here, the fundamental observation is that $F(c)/[F(c), F(c)]$ corresponds to $HC_0(F(c))$, i.e. the cyclic homology group of $F(c)$ of order zero, and the functor Π is actually the cyclic homology functor HC_0.

Therefore, the chapter starts with a condensed survey of the main definitions of cyclic homology and cohomology. For $\mathcal{A}$ an associative R-algebra, R a commutative ring, the cyclic homology of $\mathcal{A}$, $HC_*(\mathcal{A}) := \bigoplus_{n \geq 0} HC_n(\mathcal{A})$, can be defined as the homology of Connes complex $(C^{\lambda}_*(\mathcal{A}), b)$, where $C^{\lambda}_n(\mathcal{A}) := \frac{\mathcal{A}^{\otimes n+1}}{\mathrm{im}(1-t_n)}$, i.e. the cokernel of the action of $1-t_n$ onto $\mathcal{A}^{\otimes n+1} := \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (t_n is the generator of $\mathbb{Z}/(n+1)\mathbb{Z}$), and b is the *Hochschild boundary map*, i.e. the R-linear map

$$\begin{aligned} b_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := & \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ & + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

If $f : \mathcal{A} \to \mathcal{B}$ is a morphism of R-algebras, then $f_* : HC_n(f) : HC_*(\mathcal{A}) \to HC_*(\mathcal{B})$ is a morphism of R-modules. Therefore HC_n is a functor from R-**Alg**, the category of R-algebras, to R-**Mod**, the category of R-modules.

If we allow a monoidal product representation F to take values in R-**Alg**, i.e. $F(c)$ is an R-algebra, then we can consider all the cyclic homology of $F(c)$, not just $HC_0(F(c))$. Therefore, by composition with the functors HC_n, we obtain new presimplicial sets $HC_n(F(\mathbf{C}))$ (Lemma 3.1.8), which can be used to define *higher* log-functors:

Definition. A *higher logarithmic functor* of order n is a presimplicial log-additive map $\log_{[n]} : \mathcal{N}\mathbf{C} \to HC_n(F(\mathbf{C}))$, i.e. a simplicial system of maps

$$\log_{[n],x\otimes y} : \mathrm{mor}(x,y) \to HC_n(F(x\otimes y)), \quad \alpha \mapsto \log_{[n],x\otimes y}\alpha, \quad x,y \in \mathrm{obj}(\mathbf{C})$$

such that if $\alpha \in \mathrm{mor}(x,y)$ and $\beta \in \mathrm{mor}(y,z)$, then

$$\widetilde{\eta}_y(\log_{[n],x\otimes z}\beta\circ\alpha) = \widetilde{\eta}_{\otimes z}(\log_{[n],x\otimes y}\alpha) + \widetilde{\eta}_{x\otimes}(\log_{[n],y\otimes z}\beta) \in HC_n(F(x\otimes y\otimes z)).$$

All the other properties of logarithms, e.g. the logarithm of an idempotent object is trivial, follow from the case of order 0. Clearly, for $\mathbf{C} \subseteq \mathbf{Cob}_n$ a higher log-functor will be called *higher logarithmic Topological Quantum Field Theory* of dimension n. In Chapter 4 and Chapter 5 we will analyse two instances of higher LogTQFTs: a *Logarithmic Family Quantum Field Theory* (LogFQFT, i.e. when $\mathbf{C} = \mathbf{FCob}_n(B)$, the category of fibre bundles over the base space B) and a *Logarithmic Homotopy Quantum Field Theory* (LogHQFT, i.e. when $\mathbf{C} = \mathbf{HCob}_n(X)$, the category of homotopy classes of X-cobordisms, i.e. maps from a cobordism to a target space X).

Higher log-functors call for *higher traces*. Here, the idea to remember is that the R-traces on an R-algebra $\mathcal{A}$ are homomorphisms $HC_0(\mathcal{A}) \to R$. Therefore, higher traces will be homomorphisms $HC_n(\mathcal{A}) \to R$. The space of such homomorphisms, $\mathrm{Hom}(HC_n(\mathcal{A}), R)$, is in relationship with the cyclic cohomology $HC^*(\mathcal{A}) := \bigoplus_{n\geq 0} HC^n(\mathcal{A})$, of which we recalled the definition: it is the homology of the complex $(C^n_\lambda(\mathcal{A}), \beta)$, where $C^n_\lambda(\mathcal{A})$ is the sub-module of linear functionals $f \in \mathrm{Hom}(\mathcal{A}^{\otimes n+1}, R)$ such that $f(a_0\otimes\cdots\otimes a_n) = (-1)^n f(a_n\otimes a_0\otimes\cdots\otimes a_{n-1})$, and $\beta : C^n_\lambda(\mathcal{A}) \to C^{n+1}_\lambda(\mathcal{A})$ is

$$\begin{aligned}\beta(f)(a_0\otimes\cdots\otimes a_{n+1}) :=& \sum_{i=0}^{n}(-1)^i f(a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n)\\ &+(-1)^{n+1} f(a_{n+1}a_0\otimes a_1\otimes\cdots\otimes a_n).\end{aligned}$$

Kronecker pairing $HC^n(\mathcal{A}) \times HC_n(\mathcal{A}) \to R$ defines a map

$$HC^n(\mathcal{A}) \to \mathrm{Hom}(HC_n(\mathcal{A}), R),$$

which is an isomorphisms when R is a field. Therefore, by pairing with cyclic cohomology we could generalize the concept of monoidal product representation, categorical trace and log-determinant functor (Definition 3.1.13 and following). The latter will generalize to *higher log-determinants* and some manifold invariants, such as Novikov's higher signatures, will be described as such (Chapter 5).

However, passing to cyclic homology is not the only 'abelianization' method. In fact, for a ring R there exists also a projection onto its *Grothendieck group* $K_0(R)$, which factors the 'universal' trace $R \to R/[R,R]$, and a trace morphism $\tau : K_0(R) \to \frac{R}{[R,R]}$, called *Hattori-Stallings trace map*, which turns out to be a *Chern character*. Therefore, we showed that in some circumstances a higher log-functor will arise from a *universal* log-functor. In order to give its definition, and show its well-posedness, we recalled the construction of the Grothendieck group of a ring R, which in practice defines a covariant functor $K_0 : \mathbf{Ring} \to \mathbf{AbGrp}$. By functoriality, the presimplicial structure of $F(\mathbf{C})$ pushes down to $K_0(F(\mathbf{C}))$, which becomes the desired target space for:

DEFINITION. A *universal logarithmic functor* is a presimplicial log-additive map $u\text{-}\log : \mathcal{N}\mathbf{C} \to K_0(F(\mathbf{C}^*))$, i.e. a simplicial system of maps

$$u\text{-}\log_{x\otimes y} : \mathrm{mor}(x,y) \to K_0(F(x\otimes y)), \quad \alpha \mapsto u\text{-}\log_{x\otimes y}\alpha, \quad x,y \in \mathrm{obj}(\mathbf{C})$$

such that if $\alpha \in \mathrm{mor}(x,y)$ and $\beta \in \mathrm{mor}(y,z)$, then

$$\widetilde{\eta}_y(u\text{-}\log_{x\otimes z}\beta\circ\alpha) = \widetilde{\eta}_{\otimes z}(u\text{-}\log_{x\otimes y}\alpha) + \widetilde{\eta}_{x\otimes}(u\text{-}\log_{y\otimes z}\beta) \in K_0(F(x\otimes y\otimes z)).$$

Clearly, if $\mathbf{C} \subseteq \mathbf{Cob}_n$, then we call it a *universal Logarithmic Quantum Field Theory* of dimension n.

A universal log-functor yields a higher log-functor when composed with a suitable Chern character $\mathrm{ch}_n : K_0(\mathcal{A}) \to HC_{2n}(\mathcal{A})$, which in turns can be considered as a trace , i.e. an homomorphism on the abelianization of $\mathcal{A}$ taking values into an abelian group. In fact, the Chern character is a natural transformation of functors $K_0 \to HC_*$, which can be defined as $\mathrm{ch}_n([e]) := \mathrm{tr}(c(e))$ in its full generality, where $\mathcal{A}$ is a (non necessarily commutative) R-algebra, $tr : M_r(\mathcal{A})^{\otimes n} \to \mathcal{A}^{\otimes n}$ is the *generalized trace map*, and

$$c(e) := (y_n, z_n, y_{n-1}, z_{n-1}, \dots, y_1) \in M_r(\mathcal{A})^{\otimes 2n+1} \oplus M_r(\mathcal{A})^{\otimes 2n} \oplus \cdots \oplus M_r(\mathcal{A}),$$

with $y_i := (-1)^i \frac{(2i)!}{i!} e^{\otimes 2i+1}$ and $z_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} e^{\otimes 2i}$. We remark in that context that this definition reduces to the classical Chern character *à la Chern-Weil* when $\mathcal{A}$ is commutative. In particular, if $\mathcal{A} = C^\infty(B)$, B smooth manifold, then $K_0(C^\infty(B)) \cong K^0(B)$, the *topological K-theory*, $HC_*(C^\infty(B)) \cong H^*(B,\mathbb{C})$ by de Rham Theorem and ch_* is identified with the usual ring homomorphism $K^0(B) \to H^*(B,\mathbb{C})$.

In conclusion, we remarked that an algebra $\mathcal{A}$ must have some additional structure for its cyclic homology (and a Chern character) to be interesting. For instance, the cyclic homology and cohomology of a C^*-algebra can be quite poor: for example, $HC^n(C(M)) = HC^0(C(M))$ if n is even and $HC^n(C(M)) = 0$ when n is odd. This will motivate some of the choices of Chapters 4 and 5 (i.e. the *smoothing* of the index).

Chapter 4:

Here we generalize to fibre bundles, i.e. surjective surjections of manifolds $\mathcal{X} \overset{X}{\to} B$, the result of [**72**] for the topological signature. We begin by recalling the basic definitions for fibre bundles and for smooth families of vector bundles $\mathcal{E} \to \mathcal{X}$ (such as the *vertical cotangent bundle* $T^*_\pi\mathcal{X} := \bigcup_{b\in B} T^*_b X$ of a fibre bundle). In particular, $\mathcal{E} \to \mathcal{X}$ corresponds to an infinite-dimensional smooth Fréchet bundle $\pi_*(\mathcal{E}) \to B$ with fibre $\pi_*(E) := \pi_*(E_{|X_b}) = C^\infty(X_b, E_{|X_b})$. Its space of sections, $C^\infty(B, \pi_*(\mathcal{E}))$, corresponds to $C^\infty(\mathcal{X}, \mathcal{E})$, a $C^\infty(B)$-module, with which we will work in general. $C^\infty(B, \pi_*(\mathcal{E}))$ generalizes to

$$\mathcal{A}^k(B, \pi_*(\mathcal{E})) := C^\infty(\mathcal{X}, \pi^*\Lambda^k(B) \otimes \mathcal{E}),$$

the de Rham complex of smooth k-forms on B with values in $\pi_*(\mathcal{E})$. Analogously, for two smooth families $\mathcal{E} \overset{E}{\to} \mathcal{X}$, $\mathcal{F} \overset{F}{\to} \mathcal{X}$ there is a well defined smooth family of vector bundles $\Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}) \to B$ with fibre $\Psi^m(X; E, F) := \Psi^m(X_b; E_b, F_b)$. Hence, a *smooth family of ψdos of order m* (or *vertical* ψdo) associated to a fibre bundle $\mathcal{X}$ is a smooth section $\mathcal{T} \in C^\infty(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F})) = \Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F})$ and its symbol's domain is $T^*_\pi\mathcal{X}$. We will sometimes write $\mathcal{T} = (T_b)_{b\in B}$ because locally every vertical ψdo is of the form $T_b : C^\infty(X_b, E_{|X_b}) \to C^\infty(X_b, F_{|X_b})$. Clearly, vertical ψdos are the zeroth order space of a de Rham complex of ψdo-valued smooth B-forms $\mathcal{A}(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}))$. For our goal, let us consider families of Dirac operators $\mathscr{D} = (\eth_b)_{b\in B} \in \Psi^1_{\mathrm{vert}}(\mathcal{X}, \mathcal{E})$ on fibre bundles with even dimensional fibres (a vertical metric is assumed).

If $\mathcal{Y} := \partial\mathcal{X} = \emptyset$, then $\mathscr{D}$ is Fredholm and there is a well defined index class $\mathrm{ind}\mathscr{D} \in K^0(B)$. Otherwise, if $\mathcal{Y} \neq \emptyset$, we assume a product structure near $\mathcal{Y}$, so that we have the decomposition $\mathscr{D}_{|\mathcal{U}} = \Upsilon\,(\partial_t + \mathscr{D}_\mathcal{Y})$. Then the main ingredients of EBVPs are well-defined in this family case as well and yield a family of Calderón projections $\mathscr{C} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}_{|\mathcal{Y}})$, which is one instance of well-posed boundary conditions, in this case represented by *spectral sections* of $\mathscr{D}_\mathcal{Y}$, i.e. $\mathscr{P} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}_{|\mathcal{Y}})$ such that P_b is a finite rank perturbation of the APS projection $\Pi_b := \Pi_{\geq 0, b}$ for each $b \in B$. Then, with spectral sections we basically have the same results of

EBVPs for the single operator case, and the realization $\mathscr{D}_{\mathscr{P}}$ has a well-defined index class; in particular, $\mathrm{ind}\mathscr{D}_{\mathscr{P}} = \mathrm{ind}\mathscr{P}\mathscr{C} = [\mathscr{C} - \mathscr{P}] \in K^0(B)$ and quasi-additivity with respect to gluing of fibre bundles hold. Notice that spectral sections exists if and only if $\mathrm{ind}\mathscr{D}_{\mathcal{Y}} = 0$, which is the case by cobordisms invariance.

Therefore, we can define a LogFQFT, i.e. a higher LogTQFT, as follows. We consider the category $\mathbf{FCob}_n(B) \subset \mathbf{Cob}_n$ of cobordims fibered over B with fibre dimension n (an analogous category is used to define Fibered QFTs in [**80**]) and the representation $F^{-\infty}_{\mathrm{vert}}(\mathcal{Y}) := \Psi^{-\infty}_{\mathrm{vert}}(\mathcal{Y}, \Lambda_\pi(\mathcal{Y}))$, with $\Lambda_\pi(\mathcal{Y}) := \Lambda(T_\pi\mathcal{Y}) \to \mathcal{Y}$ the bundle of vertical forms. For $\mathcal{X} \in \mathrm{mor}_{\mathbf{FCob}_{2n}(B)}(\mathcal{M}_0, \mathcal{M}_1)$, we consider its family signature operator $\mathscr{D}^{\mathrm{Sign}}$, together with a particular kind of spectral section $\mathscr{P}$ called *symmetric*. Symmetric spectral sections were defined by [**43**], upon meeting some sufficient condition, and are what is needed to have a homotopy invariant index class $\mathrm{ind}\mathscr{D}^{\mathrm{Sign}}_{\mathscr{P}}$. Then we define a universal LogTQFT $u\text{-}\log^{\mathrm{Sign}} : \mathcal{N}\mathbf{FCob}_n(B) \to K_0(F^{-\infty}_{\mathrm{vert}}(\mathbf{FCob}_n(B))) \otimes \mathbb{Q}$ by setting

$$u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0 \sqcup \mathcal{M}_1} \mathcal{X} := \widetilde{\phi}_{\sharp, \mathcal{M}_0 \sqcup \mathcal{M}_1}\left([\mathscr{C} - \mathscr{P}]\right) \in K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0 \sqcup \mathcal{M}_1)) \otimes \mathbb{Q},$$

with $\widetilde{\phi}_{\sharp, \mathcal{M}_0 \sqcup \mathcal{M}_1}$ the canonical isomorphism $K_0(F^{-\infty}_{\mathrm{vert}}(\partial\mathcal{X})) \cong K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0 \sqcup \mathcal{M}_1))$. Since $\mathscr{P}$ is symmetric, log-additivity follows in $K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0 \sqcup \mathcal{M}_1 \sqcup \mathcal{M}_2)) \otimes \mathbb{Q}$

$$\widetilde{\eta}_{\mathcal{M}_1} \log^{\mathrm{Sign}}_{\mathcal{M}_0 \sqcup \mathcal{M}_2} \mathcal{X}_1 \cup \mathcal{X}_2 = \widetilde{\eta}_{\mathcal{M}_2} \log^{\mathrm{Sign}}_{\mathcal{M}_0 \sqcup \mathcal{M}_1} \mathcal{X}_1 + \widetilde{\eta}_{\mathcal{M}_0} \log^{\mathrm{Sign}}_{\mathcal{M}_1 \sqcup \mathcal{M}_2} \mathcal{X}_2.$$

Now, $F^{-\infty}_{\mathrm{vert}}(\mathcal{M})$ is shown to be Morita equivalent to $C^\infty(M)$. Therefore, $K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M})) = K^0(B)$ and the Chern character ch_*, which in this case corresponds to the classical one $K^0(B) \to H^{2*}(B)$, yields the higher LogTQFT

$$\log^{\mathrm{Sign}}_{\mathcal{M}_0 \sqcup \mathcal{M}_1} \mathcal{X} = \mathrm{ch}_*(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0 \sqcup \mathcal{M}_1} \mathcal{X}) = \mathrm{ch}_*\left([\mathscr{C} - \mathscr{P}]\right) \in H^{2*}(B),$$

which is equal to $\mathrm{ch}([\mathscr{C} - \mathscr{P}]) = \sum_{k=0}^{\dim B} \frac{1}{k!} \mathrm{Tr}_{\mathcal{Y}/B}\left(R^k_{\mathscr{C}} - R^k_{\mathscr{P}}\right)$, where $\mathrm{Tr}_{\mathcal{Y}/B}$ is integration along the fibres and $R_{\mathscr{P}}$ are curvatures. In particular, for $k = 0$ we have the topological signature of the fibre X. In conclusion, we obtain higher traces by pairing with cyclic cohomology. For instance, we can obtain the signature of the total space $\mathcal{X}$.

Chapter 5:

This chapter describes another higher LogTQFT, similar to the family one but belonging to the noncommutative geometry setting. The category we are working with is the one of *homotopy cobordisms* $\mathbf{HCob}_n(X) \subset \mathbf{Cob}_n$, i.e. maps $r : M \to X$ and homotopy classes of maps of cobordisms between them, as described in [**66**].

As anticipated, a higher log-functor representation of this category can be called LogHQFT, from Homotopy Quantum Field Theory (which indeed can induce, as much as a LogTQFT can induce a TQFT). This is the category to which a Galois Γ-covering $\widetilde{M} \to M$ belongs. Thus, we start the chapter by recalling that a covering is called *Galois* if Γ is discrete and finitely presented and acts on the fibres freely and transitively. Such coverings are principal Γ-bundle, thus isomorphism classes are in bijective correspondence with homotopy classes of continuous maps into the *classifying space* of Γ, i.e. $r : M \to B\Gamma$.

If $\eth$ is a Dirac operator on M, then we recall that it is possible to associate to a Galois covering $r : M \to B\Gamma$ a twisted Dirac $\mathscr{D}_{(M,r)}$ in a standard way. Such Dirac operator falls into the Mishchenko-Fomenko ψdodifferential calculus: in fact, it is a $C^*_r\Gamma$-linear operator on the *Hilbert module* $H^1_{C^*_r\Gamma}(M, E \otimes \mathcal{V})$, with $C^*_r\Gamma$ the *reduced* (noncommutative) C^*-algebra associated to Γ and $\mathcal{V}$ a flat bundle of coefficients associated to the covering. Definitions and a description of the construction are given in the section.

Again, let us restrict to the case $\dim M$ even. If $\partial M = \emptyset$, then $\mathscr{D}_{(M,r)}$ has a well defined index class in $K_0(C^*_r\Gamma)$; otherwise, once again we must impose boundary conditions via spectral sections. In fact, if $\partial M \neq \emptyset$, suitably defined spectral sections exist by cobordism invariance and define an index class $\operatorname{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) \in K_0(C^*_r\Gamma)$. However, in order to have interesting cyclic homology and a 'good' Chern character, we need to restrict to a *smooth* subalgebra $\mathcal{B}$, i.e. a subalgebra of $C^*_r\Gamma$ which is dense and closed under holomorphic functional calculus. This process, called *smoothing* of the index, in fact does not change the K-theory, since for a subalgebra with this properties $K_0(C^*_r\Gamma) = K_0(\mathcal{B})$. But now, for a spectral section to be chosen in the proper algebra $\Psi^0_{\mathcal{B}}(\partial M, (E \otimes \mathcal{V})_{|\partial M})$, the group Γ must have some additional structure. It will suffice that Γ is *virtually nilpotent*, i.e. it contains a nilpotent subgroup of finite index, which we will assume from this moment on.

In this Hilbert module context, though, there is a nuisance to cope with: a formula $\operatorname{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) = \operatorname{ind}\mathscr{P}\mathscr{C}$ is yet to be proved, even if the existance of a Calderón projection has been confirmed ([**1**]). Hopefully, the information carried by the index still comes from the boundary, as in the classical case.

Therefore, we define a universal LogTQFT in a way similar to the one of Chapter 4. We consider the a representation $F^{-\infty}_\Gamma(M, r) := \Psi^{-\infty}_{\mathcal{B}}(M, \Lambda(M) \otimes \mathcal{B})$, $(M, r) \in \operatorname{obj}(\mathbf{HCob}_{2n}(B\Gamma))$, and the twisted signature operator $\mathscr{D}^{\mathrm{Sign}}_{(M,r)}$ associated to $(W, F) \in \operatorname{mor}_{\mathbf{HCob}_{2n}}((M_0, r_0), (M_1, r_1))$. We will need to consider conditions similar to the one of the family case in order to have symmetric spectral sections $\mathscr{P} \in \Psi^0_{\mathcal{B}}$, which will yield an homotopy invariant index class. Thus, we define $u\text{-}\log^{\mathrm{Sign}} : \mathcal{N}\mathbf{HCob}_n(B\Gamma) \to K_0(F_\Gamma^{-\infty}(\mathbf{HCob}_n(B\Gamma)))$ as

$$u\text{-}\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)}(W,F) := \widetilde{\phi}_{\sharp,(M_0,r_0)\sqcup(M_1,r_1)}(\mathrm{ind}(\mathscr{D}_{(W,F)},\mathscr{P}))$$

and since $K_0(F_\Gamma^{-\infty}((M_0,r_0)\sqcup(M_1,r_1))) = K_0(\mathcal{B})$ by Morita equivalence, we can consider the Chern character $\mathrm{ch}_* : K_0(\mathcal{B}) \to HC_{2*}(\mathcal{B})$ and obtain a LogHQFT by composition:

$$\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)}(W,F) = \mathrm{ch}_*(\mathrm{ind}(\mathscr{D}_{(W,F)},\mathscr{P})) \in HC_{2*}(\mathcal{B}).$$

Again, log-additivity follows from index additivity with respect to gluing when symmetric spectral sections are considered (a feature that still holds in this setting). Finally, pairing with cyclic cohomology will define higher traces which will yield *Novikov higher signatures*. These scalars, in this case homotopy invariants, are defined in the following way: since Γ is virtually nilpotent, for $[c] \in H^*(\Gamma,\mathbb{C})$ there exists an associated $\varphi_c \in HC^*(\mathcal{B})$; then a higher signature is the quantity:

$$\begin{aligned}\mathrm{Sign}(W,F;[c]) :=& \langle \mathrm{ch}_*(\mathrm{ind}(\mathscr{D}_{(W,F)},\mathscr{P})),\varphi_c\rangle \\ =& \tau^c_{(M_0\sqcup M_1,s_1\sqcup s_2)}\left(\log^{\mathrm{Sign}}_{(M_0\sqcup M_1,s_1\sqcup s_2)}(\overline{W},r)\right).\end{aligned}$$

Their additive property will clearly follow as a consequence of log-additivity.

Chapter 6:

This last chapter forms a separate Part by itself. Although related to the leitmotif of Part I and Part II, it is mostly focused on torsion invariants of manifolds. In some cases, it will be possible to characterize them as trace-characters of LogTQFT. The main object of study will be the residue analytic torsion of a manifold X (with or without boundary). Its construction originates from observing that the analytic torsion, which is really the analytic 'twin' of the Reidemeister torsion, can equivalently be described as a quasi-trace-character. In order to define our object, and to make these statements more precise, we recall at the beginning of the chapter the main definitions and properties of Reidemeister torsion (from now on R-torsion) and analytic torsion.

For the R-torsion, the starting point is a C^1-triangulation of X, i.e. a CW complex (which we can call with the same letter) $X = \bigcup_{r=0}^n \bigcup e^r$, $e^r \subset X$ an r-cell, with universal cover $\widetilde{X} = \bigcup_{g\in\pi_1(X)} \bigcup_{r=0}^n \bigcup g\tilde{e}^r$. Let $X^{(r)} = \bigcup_{j\leq r} \bigcup e^r$ be the r-skeleton of X, with induced cover $\widetilde{X}^{(r)}$. Then, the relative homology module $C_r(\widetilde{X}) := H_r(\widetilde{X}^{(r)}, \widetilde{X}^{(r-1)})$ defines a chain complex of finitely generated free $\mathbb{R}[\pi_1(X)]$-modules, where $\mathbb{R}[\pi_1(X)]$ is the group ring of finite formal sums $\sum_k \alpha_k g_k$, for $\alpha_k \in \mathbb{R}$ and $g_k \in \pi_1(X)$. For the R-torsion to be a topological invariant (i.e. invariant modulo homeomorphisms), the complex $C_*(\widetilde{X})$ should be acyclic, i.e. its homology should be trivial. Since it is not the case, we can fix this by tensoring with $\mathbb{R}^N$, which can be seen as a $\mathbb{R}[\pi_1(X)]$-module via a homomorphism $\rho : \pi_1(X) \to O(N)$, called *orthogonal representation*. The new complex of finite dimensional vector spaces $C_r(X,\rho) := \mathbb{R}^N \otimes_{\mathbb{R}[\pi_1(X)]} C_r(\widetilde{X})$ can be made acyclic for suitable choices of ρ. Such chain complex has a boundary operator d, induced by the natural one of the CW complex, which can be represented by a real matrix after choosing a basis for $C_r(X,\rho)$. Therefore, the (logarithm of the) R-torsion of X can be defined in this context as the scalar quantity:

$$\log \tau_X(\rho) = \frac{1}{2}\sum_{r=0}^{n}(-1)^{r+1} r \log\det \Delta_r^c,$$

where $\Delta_r^c := d_{r+1}d_{r+1}^* + d_r^* d_r : C_r(X,\rho) \to C_r(X,\rho)$ is called *the combinatorial Laplacian*, d^* being the transpose of d. We notice that it is well-defined, as acyclicity of $C_*(X,\rho)$ makes Δ_r^c invertible. Now, $\log\det\Delta_r^c$ can be expressed in terms of $\zeta_r^c(s)$, the *zeta function* of Δ_r^c. This can be defined as the meromorphic extension of $\sum_{\lambda_i>0}\lambda_i^{-s}$, λ_i the eigenvalues of Δ_r^c, which is holomorphic at $s=0$. Then, $\log\det\Delta_r^c = -\frac{d}{ds}\zeta_r^c(0)$ and $\log\tau_X(\rho) = \frac{1}{2}\sum_{r=0}^{n}(-1)^r r \frac{d}{ds}\zeta_r^c(0)$.

This characterization of R-torsion was the starting point for Ray and Singer, [**65**], to define analytic torsion. In fact, out of the metric of a closed manifold X we can define the (twisted) *Hodge-Laplacian* $\Delta_k : \Omega^k(X,E_\rho) \to \Omega^k(X,E_\rho)$, where E_ρ is a flat bundle associated to ρ, which can be used to make $\Omega(X,E_\rho)$ acyclic. Since Δ_k is elliptic and self-adjoint on a closed manifold, it has only countably many positive eigenvalues and we can consider the sum $\sum_{\lambda_i>0}\lambda_i^{-s}$ of complex powers of its eigenvalues, as for the combinatorial Laplacian. Then, its meromorphic extension, the *zeta function* $\zeta_k(s) := \zeta(\Delta_k, s)$, is holomorphic at $s=0$ and Ray and Singer defined the (logarithm of the) *analytic torsion* as:

$$\log T_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k k \frac{d}{ds}\zeta_k(0).$$

Along the way, they obtained a regularized determinant of Δ_k, the *ζ-determinant* $\det_\zeta \Delta_k := \exp(-\frac{d}{ds}\zeta_k(0))$, yielding $\log T_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} k \log\det_\zeta \Delta_k$.

Around 1980, Cheeger and Müller separately proved that R-torsion and analytic torsion of a closed manifold coincide, and indeed already Ray and Singer proved that they share similar properties, e.g. they are both trivial on even-dimensional manifolds, and $T_X(\rho)$ is a smooth invariant for $\Omega(X,E)$ acyclic. Moreover, for a path of metrics $u \mapsto g^X(u)$ and $*_k = *_k(u)$ the Hodge operator associated to $g^X(u)$,

$$\frac{d}{du}\log \tau_X(\rho) = \frac{d}{du}\log T_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k \text{ tr } \alpha_{k|_{\ker \Delta_k}}, \qquad \alpha_k := *_k^{-1} \dot{*}_k$$

Our starting point to define residue torsions is the observation that $\log\det \Delta_k^c = \operatorname{tr}\log\Delta_k^c$, where, by holomorphic functional calculus,

$$\log \Delta_k^c := \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda \ (\Delta_k^c - \lambda)^{-1} d\lambda,$$

$\mathscr{C}$ a keyhole path enclosing $\operatorname{spec}(\Delta_k^c)$. Thus, $\log \tau_X(\rho)$ becomes a tr-character of the logarithm $\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}k\log\Delta_k^c$. In a similar way, we have

$$\log \Delta_k = \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda \ (\Delta_k - \lambda)^{-1} d\lambda$$

and we can show that $\log\det_\zeta \Delta_k = \mathrm{TR}_\zeta \log\Delta_k$, where TR_ζ is the *Kontsevich-Vishik quasi-trace*, the extension of the classical trace to $\Psi^{\mathbb{Z}}$ with respect to the complex power gauging. Therefore, the analytic torsion is the TR_ζ-character of a *torsion logarithm* $\mathbb{T}_X(\rho) = \frac{1}{2}\bigoplus_{k=0}^{n}(-1)^{k+1}k\log\Delta_k \in \Psi^{\leq 0}(X, \Lambda(X)\otimes E_\rho)$.

Therefore we can consider a *generalized* torsion logarithm

$$\mathbb{T}_X^\beta(\rho) := \frac{1}{2}\bigoplus_{k=0}^{n}(-1)^{k+1}\beta_k \ \log\Delta_k, \qquad \beta = (\beta_0, \dots, \beta_n) \in \mathbb{R}^{n+1}$$

and investigate other possible invariants of X originating as its trace-characters. Now, every trace functional $\tau : \Psi^{\leq 0}(X,F) \to \mathbb{C}$ must be a linear combination of a *leading symbol trace* $\tau_{u,0}(A)$, defined from the trace of the leading term of the asymptotic expansion of the symbol, $\operatorname{tr}\ \sigma^A(x,\xi) \in C^\infty(S^*X)$, via pairing with a distribution $u \in \mathcal{D}'(S^*X)$, and Wodzicki's *residue trace*

$$\operatorname{res}(A) := \int_X \left(\int_{|\xi|=1} \operatorname{tr}\ \sigma^A_{-\dim X}(x,\xi)\ d_\xi S\right) dx,$$

which originates from the $-n$ term in the asymptotic expansion of the symbol (and is the unique trace on the algebra $\Psi^{\mathbb{Z}}(X,F)$). Therefore, we consider these two traces and study the associated trace-characters.

Composing $\mathbb{T}_X^\beta(\rho)$ with the leading symbol trace yields the exotic analytic torsion $\log T_X^{\mathrm{lead},\beta,u}(\rho) := \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \tau_{0,u}\log\Delta_k$, which however turns out to be identically zero for each $\beta \in \mathbb{R}^{n+1}$ and $u \in \mathcal{D}'(S^*X)$. On the other hand, the *residue analytic torsion*

$$\log T_X^{\mathrm{res},\beta}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \ \mathrm{res}\log\Delta_k$$

turns out to be more interesting and a smooth invariant for specific values of β. In fact, by considering the derivative $\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho)$ with respect to a smooth family of metrics $u \in \mathbb{R} \to g^X(u)$, and exploiting Scott's formula

$$(\star\star) \qquad -\frac{1}{2}\mathrm{res}\log\Delta_k = \zeta_k(0) + \dim\ker\Delta_k,$$

we proved the following classification theorem (Theorem 6.2.28):

THEOREM. If n is odd, then $\log T_X^{\mathrm{res},\beta}(\rho) = 0 \ \forall\beta \in \mathbb{R}^{n+1}$. If n is even, $\log T_X^{\mathrm{res},\beta}(\rho)$ is a smooth invariant if and only if β equals:

$$\underline{1} := (1,\ldots,1) \quad \text{or} \quad \underline{\omega} := (0,1,\ldots,n).$$

The corresponding residue analytic torsions are equal, respectively, to the Euler characteristic χ and the *derived Euler characteristics* χ':

$$\log T_X^{\mathrm{res},\underline{1}}(\rho) = \chi(X,E_\rho) \qquad \text{and} \qquad \log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \chi'(X,E_\rho).$$

In particular, for a smooth path of metrics $u \in \mathbb{R} \to g_X(u)$ we have:

$$\frac{d}{du}\log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} \ \underbrace{\mathrm{res}(\alpha_k)}_{=0},$$

and even if it vanishes, it analogous to the derivative of the analytic torsion.

The derived Euler characteristic $\chi'(X) := \sum_{k=0}^{n}(-1)^k k \dim H^k(X)$ is another topological invariant and equals $\frac{n}{2}\chi(X)$ when $n = \dim X$ is even. Therefore, in conclusion $\log T_X^{\mathrm{res},\beta}(\rho)$ is a smooth invariant if and only if it is a homotopy invariant, in which case coincides with either $\chi(X)$ or $\frac{n}{2}\chi(X)$. This also yields that the torsion logarithms $\mathbb{T}_X^{\underline{1}}(\rho)$ and $\mathbb{T}_X^{\underline{\omega}}(\rho)$ are also invariants of X. Finally, using $(\star\star)$ again, and a strategy similar to **[65]**, we showed that a *generalized* analytic torsion

$$\log T_X^{\beta}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \log\det{}_\zeta\Delta_k$$

is a smooth invariant if and only if β is again equal to $\underline{1}$ or $\underline{\omega}$.

LogTQFT can provide a functorial setting also for torsion invariants, in the following way. The representation $F_{\mathbb{Z}}(M) := \Psi^{\mathbb{Z}}(M,\Lambda(M))$, $M \in \mathrm{obj}(\mathbf{Cob}_n)$ is unoriented and, as we said, has a trace: the noncommutative residue res. Thus, for $\overline{X} \in \mathrm{mor}(M_0, M_1)$, $\partial X = Y_0^- \sqcup Y_1$, we can define a LogTQFT by setting

$$\log^{\beta}_{M_0 \sqcup M_1} \overline{X} := \pi_{M_0 \sqcup M_1} \circ \kappa_{\sharp} \left(\frac{1}{2} \bigoplus_{k=0}^{n} (-1)^k \beta_k \log \Delta_{k,Y_0} \oplus \frac{1}{2} \bigoplus_{k=0}^{n} (-1)^{k+1} \beta_k \log \Delta_{k,Y_1} \right),$$

with character, for $\beta = \underline{\omega}$, $\mathrm{res}\left(\log^{\underline{\omega}}_{M_0 \sqcup M_1} \overline{X}\right) = \chi'(M_1) - \chi'(M_0)$. In this case, log-additivity is straightforward. Additionally, if we restrict to the category of h-cobordisms h-$\mathbf{Cob}_n$, where the objects are deformation retracts of the cobordisms, we can characterize the analytic torsion as the TR_{ζ}-character of the same LogTQFT. The res-character in this context is, by homotopy invariance, trivial.

The same results can actually be reproduced for a fibre bundle with closed fibre $\mathcal{X} \overset{X}{\to} B$. On the one hand, the de Rham operator $d^{\mathcal{X}} + \delta^{X}$ associated to the total space $\mathcal{X}$ is a superconnection adapted to a family of de Rham operators $(d^{X_b} + \delta^{X_b})_{b \in B}$. On the other hand, the Laplacian $\Delta^{\mathcal{X}}$, i.e. the curvature of the superconnection, is adapted to a family of Laplacians $(\Delta^{X_b})_{b \in B}$. Since logarithm and residue torsion are well-defined for families of differential operators and superconnection, with suitable generalizations, we were able to define a family torsion logarithm and *family residue analytic torsion*

$$\mathscr{T}^{\mathrm{res},\beta}_{\mathcal{X}} = \frac{1}{2} \sum_{k=0}^{\dim X} (-1)^{k+1} \beta_k \,\mathrm{res} \log \Delta_k^{\mathcal{X}} \in H^*(B, \mathbb{R})$$

and show that the same result of the single operator case holds also for fibre bundles. The difference here is that $\mathscr{T}^{\mathrm{res},\beta}_{\mathcal{X}}$ for $\beta = \underline{1}$, resp. $\beta = \underline{\omega}$, equals $\chi(X)$, resp. $\frac{\dim X}{2} \chi(X)$, where X is the fibre, since the cohomology bundle $H(X, E) \to B$ is flat. Here, we also use the family torsion logarithm to define a 'simple' LogFQFT.

We conclude the chapter with the appropriate generalization to a manifold X with boundary Y. Since the analytic torsion is defined in terms of the eigenvalues of the Laplacian Δ_k on k-forms, we need self-adjoint boundary conditions, which are once again represented by the relative (or the absolute) ones. For instance, $\Delta_{k,\mathcal{R}}$ stands for the Laplacian on $\Omega^k(X, E_\rho)$ with *relative boundary conditions*, i.e. $\mathcal{R}\gamma\omega = 0$ and $\mathcal{R}\gamma\delta\omega = 0$. When such boundary conditions are imposed, the Laplacian has a spectrum of discrete non-negative eigenvalues accumulating at infinity, as in the case $Y = \emptyset$. Therefore, one can define a logarithm

$$\log \Delta_{k,\mathcal{R}} := \lim_{s \searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}} \log \lambda \lambda^{-s} \; (\Delta_{k,B} - \lambda)^{-1} d\lambda,$$

([**22**]) and a $\operatorname{res}\log\Delta_{k,\mathcal{R}}$ via the generalization to Boutet de Monvel operators of the residue trace ([**21**]). Thanks to this, we obtain a *relative residue analytic torsion*

$$\log T^{\mathrm{res},\beta}_{X,\mathcal{R}}(\rho)=\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \operatorname{res}\log\Delta_{k,B}.$$

We are then able to generalize the classification theorem of the closed case:

THEOREM. Let X be an oriented manifold with boundary Y. Then $\log T^{\mathrm{res},\beta}_{X,B}(\rho)$ is a smooth invariant if and only if $\beta=\underline{1}$ or $\beta=\underline{\omega}$. The corresponding residue analytic torsions are:

$$\log T^{\mathrm{res},\underline{1}}_{X,\mathcal{R}}(\rho)=\chi(X,Y,E_\rho)\quad\text{ and }\quad\log T^{\mathrm{res},\underline{\omega}}_{X,\mathcal{R}}(\rho)=\chi'_B(X,Y,E_\rho)+\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(0).$$

In particular, for a smooth path of metrics $[0,1]\ni u\mapsto g^X(u)$ for which the normal direction to the boundary is the same, we have:

$$\frac{d}{du}\log T^{\mathrm{res},\underline{\omega}}_{X,\mathcal{R}}(\rho)=\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\ \operatorname{res}\ \alpha_k.$$

We remark that the term $\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(0)$ does not vanish as in the closed manifold case (some examples are provided), but $\sum_{k=0}^{n}(-1)^k\zeta_{k,\mathcal{R}}(s)$ is zero also in this case and is responsible for the equality $\log T^{\mathrm{res},\underline{1}}_{X,\mathcal{R}}(\rho)=\chi(X,Y,E_\rho)$. The proof is analogous to the closed case and uses a generalization of Scott's formula ($\star\star$) to the boundary case, found in [**27**]. We conclude this final chapter by showing quasi-additivity of the residue torsion (Theorem 6.5.17):

$$\log T^{\mathrm{res},\beta_1}_{X,R}(\rho)=\log T^{\mathrm{res},\beta_1}_{X_1,R}(\rho)+\log T^{\mathrm{res},\beta_1}_{X_2,R}(\rho)+\log T^{\mathrm{res},\beta_1}_{Y}(\rho)+\frac{1}{2}\chi(Y)$$

and remark that $\chi'(X)=\chi'(X,Y)+\chi'(Y)+\frac{1}{2}\chi(Y)$ if $\dim X$ is odd, but not when it is even (a counterexample will be provided).

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CHAPTER 0

Background

In this introductory chapter, we set the notation and recall some standard results that will be given for granted in the sequel. The first section recalls the basic set up we will be working with. In the second one, we set the notation for pseudodifferential operators and their symbols. In the third one, we describe the decomposition of a smooth form, over the boundary of a manifold, into a *tangential* and a *normal* component. Finally, the fourth section recalls the Euler characteristic of a manifold and its properties.

0.1. Riemannian manifolds with boundary and restriction of sections

Let X be an n-dimensional *manifold*, i.e. from now on a compact C^∞-manifold, possibly with non-empty smooth boundary $Y := \partial X$. If $Y = \emptyset$, we will say that X is *closed*. If X is also oriented, then Y inherits a coherent orientation from X. When X is considered with the opposite orientation, we will write X^-. For $x \in X$, let T_xX and T_x^*X denote the tangent and cotangent spaces of X at x, respectively, and TX and T^*X its tangent and cotangent bundles. For standard definitions about differentiable manifolds with boundary we refer to **[16]** and **[68]**.

For $c \in \mathbb{R}_+ := \{x \in \mathbb{R} |\ x > 0\}$, let $U := [0, c) \times Y$ be a collar neighborhood[1] of Y, where coordinates $x = (t, y)$ are chosen in such a way that $y \in Y$ and $t \in [0, c)$ corresponds to an inward normal covariant derivative, denoted by ∂_t. D_t will stand for $-i\partial_t$ as usual in the context of microlocal analysis, where $i := \sqrt{-1}$.

Let g^X denote a choice of Riemannian metric for X, and $v(x)dx$ the associated volume element. The boundary Y inherits a metric g^Y with associated volume element $v(0, y)dy$. We will consider a *product stucture* near the boundary (see for instance **[25]**), i.e. on U:

$$g^X = dt^2 + g^Y \quad \text{and} \quad v(x)dx = v(0, y)dydt. \tag{0.1.1}$$

Let $E \xrightarrow{\pi} X$ be a Hermitian vector bundle over X of rank N. We will denote by ∇^E its connection, by $C^\infty(X, E)$ the space of smooth sections of E, and by

[1] Which always exists, see for instance Theorem (1.2), **[16]**.

$H^s(X,E)$ the associated Sobolev space, i.e. its Hilbert space completion with respect to the measure $(1+|\xi|^2)^{\frac{s}{2}}d\xi$ (see for instance §11 of [**10**] for a presentation). $H^0(X,E) := L^2(X,E)$ has inner product defined by the metric on X:

$$\langle s_1, s_2\rangle := \int_X g_x\left(s_1(x), s_2(x)\right) v(x)dx \quad \text{ for } \; s_1, s_2 \in C^\infty(X,E).$$

By $\mathcal{D}'(X,E)$ we will denote the space of distributions $C^\infty(X,E) \to \mathbb{C}$.

Let $E' := E_{|Y} = \bigsqcup_{y\in Y} E_y$ be the restriction of E to the boundary. Then by product structure $E_{|U} = \iota^* E'$, where ι^* is the pull-back of the natural embedding $\iota : Y \hookrightarrow U$. The restriction to the boundary Y defines a continuous *trace map* $\gamma : C^\infty(U, E_{|U}) \to C^\infty(Y, E')$:

$$s_{|Y}(y) := (\gamma s)(y) := s(0,y), \qquad s \in C^\infty(U, E_{|U}), \tag{0.1.2}$$

which extends to a continuous and uniformly bounded operator (Corollary 11, [**10**]):

$$\gamma : H^s(X,E) \to H^{s-\frac{1}{2}}(Y,E'), \quad \text{for} \quad s > \frac{1}{2}.$$

We remark that the definition of γ can be extended to all $s \in \mathbb{R}$.

EXAMPLE 0.1.1 (§1.1, [**68**]). The restriction $TX_{|Y} := \bigcup_{y\in Y} T_yX$ of TX to Y is a classical example of restriction of a vector bundle to the boundary. If $\iota : Y \hookrightarrow X$ is the natural embedding, then $d_y\iota : T_yY \to T_yX$ is injective $\forall y \in Y$ and TY is a 1-codimensional sub-bundle of $TX_{|Y}$. In fact, by product structure $TX_{|Y} = TY \oplus \mathbb{R}$ and $d\iota$ induces a natural inclusions of the space of vector fields $\Gamma(TY) := C^\infty(Y, TY)$ into $\Gamma(TX_{|Y}) := C^\infty(Y, TX_{|Y})$. Moreover, the pull-back $\iota^* : \Gamma(TX) \to \Gamma(TY)$ is surjective.

We will assume familiarity with the concept of *gluing of manifolds* along diffeomorphic connected components of their boundaries. For example, if $Y_i := \partial X_i$, $i = 1, 2$, and $\phi : Y_1 \overset{\cong}{\to} Y_2$ is a diffeomorphism, then we write $X_1 \cup_\phi X_2$ for the closed manifold defined by the gluing. The operation can easily be generalized to some connected components of the boundaries Y_1 and Y_2. Here we only recall the *Uniqueness of Gluing Theorem*, i.e. different collar neighborhoods of the boundaries yield different but diffeomorphic manifolds. The main reference in this case is [**31**], Chapter 8, §2.

We will assume that X is embedded in a closed n-dimensional manifold $\widetilde{X}$, such that Y is smoothly embedded in $\widetilde{X}$. Then Y has a symmetric tubular neighborhood $\widetilde{U}$ in $\widetilde{X}$ such that $x = (t, y)$, with $|t| < c(y)$ and $c(y) \in \mathbb{R}^+$ (§7, [**25**]). For example, $\widetilde{X}$ could be the closed double $X \cup X_1$, where $X_1 = X$ or $X_1 = X^-$ (in case we consider oriented manifolds). Its construction is explained in §9, [**10**]. Likewise, the vector bundle E can be considered to be the restriction to X of a bundle $\widetilde{E} \to \widetilde{X}$ of rank N. As for (0.1.2), the restriction $\widetilde{\gamma} : C^\infty(\widetilde{X}, \widetilde{E}) \to C^\infty(Y, E')$ extends to a continuous and uniformly bounded operator

$$\widetilde{\gamma} : H^s(\widetilde{X}, \widetilde{E}) \to H^{s-\frac{1}{2}}(Y, E'), \quad \text{for} \quad s > \frac{1}{2},$$

which has adjoint $\widetilde{\gamma}^* : H^{-s+\frac{1}{2}}(Y, E') \to H^{-s}(\widetilde{X}, \widetilde{E})$, $(\widetilde{\gamma}^*\phi)(y,t) := \phi(y) \otimes \delta(t)$, with δ the *delta distribution* supported in Y (see §1.3, [**26**]).

0.2. Classical pseudodifferential operators and traces

Here, we only mention some basic definitions of classical pseudodifferential operators on closed manifolds and manifolds with boundary for the sake of notation. For a complete exposition, we refer to [**32**], [**79**], and [**81**] for closed manifolds, and [**26**] for manifolds with boundary.

Let X be closed and $m \in \mathbb{C}$. For a local trivialization (V, φ), let us denote by $\mathsf{S}^m := \mathsf{S}^m(V \times \mathbb{R}^n, \mathrm{End}(\mathbb{C}^N))$ the space of symbols of order m; as usual, $\mathsf{S}^{-\infty} := \bigcap_{k\in\mathbb{R}} \mathsf{S}^k$ denotes the ideal of smoothing symbols. For the space of *classical symbols* of order m, i.e. $a(x,\xi) \in \mathsf{S}^m$ such that:

$$a(x,\xi) = \sum_{j\geq 0} a_{m-j}(x,\xi) \in \mathsf{S}^m/\mathsf{S}^{-\infty} \qquad \text{and}$$

$$a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x,\xi) \quad \text{for} \quad t \geq 1, |\xi| \geq 1,$$

we will write $\mathsf{CS}^m := \mathsf{CS}^m(V \times \mathbb{R}^n, \mathrm{End}(\mathbb{C})^N)$. Notice that $\mathsf{CS} := \bigcup_{m\in\mathbb{C}} \mathsf{CS}^m$ is not a linear space , but $\mathsf{CS}^{\mathbb{Z}} := \bigcup_{k\in\mathbb{Z}} \mathsf{CS}^k$ is a Fréchet algebra (§1.5.2, [**75**]).

Let $\Psi(X,E) := \bigcup_{m\in\mathbb{C}} \Psi^m(X,E)$ denote the semigroup of *classical pseudodifferential operators* (from now on *classical ψdos*), i.e. $A : C^\infty(X,E) \to C^\infty(X,E)$ is a ψdo such that $\sigma^A(x,\xi) \in \mathsf{CS}^m$, where $\sigma^A(x,\xi) \sim \sum_{j\geq 0} \sigma^A_{m-j}(x,\xi)$ is the symbol of A and therefore $\sigma^A_m(x,\xi)$ denotes its *principal* (or *leading*) symbol. As for symbols, $\Psi(X,E)$ is only a semigroup; however, $\Psi^{\mathbb{Z}}(X,E) := \bigcup_{k\in\mathbb{Z}} \Psi^k(X,E)$ is a Fréchet algebra (§1.5.4, [**75**]).

Let $A \in \Psi^m(X,E)$; then A is *elliptic* if its principal symbol σ^A_m is an invertible section, i.e. $\sigma^A_m(x,\xi) \in \mathrm{End}(\mathbb{C}^N)$ is invertible for each $(x,\xi) \in T^*X \setminus 0$ (§1.5.3.1, [**75**]). This equivalently means that there exists $p \in \mathsf{S}^{-m}$ such that $p\sigma^A - I$ and $\sigma^A p - I$ belong to $\mathsf{S}^{-\infty}$ (§18.1, [**32**]). $\Psi_{\mathrm{Ell}}(X,E) := \bigcup_{m\in\mathbb{C}} \Psi^m_{\mathrm{Ell}}(X,E)$ denotes the space of elliptic ψdos and is a sub-semigroup of $\Psi(X,E)$. In particular, $\Psi^{\mathbb{Z}}(X,E)$

is the smallest algebra containing all elliptic differential operators and their parametrices (§1.1.8, [**75**]).

Let $A \in \Psi^m(X,E)$, $m \in \mathbb{Z}$. Then A is *odd-class* if in any local trivialization $\sigma^A_{m-j}(x,\xi) = (-1)^{m-j}\sigma^A_{m-j}(x,-\xi)$, $j \geq 0$. In this case we write $A \in \Psi^m_{(-1)}(X,E)$ and $\Psi^{\mathbb{Z}}_{(-1)}(X,E) := \bigcup_{m\in\mathbb{Z}} \Psi^m_{(-1)}(X,E)$ is a subalgebra of $\Psi^{\mathbb{Z}}(X,E)$ containing differential operators and smoothing ψdos (§7, [**39**]).

In particular, a ψdo A is called *smoothing* if $\sigma^A \in \mathsf{S}^{-\infty}$. In this case, we write $A \in \Psi^{-\infty}(X,E) := \bigcap_{m\in\mathbb{R}} \Psi^m(X,E)$ and $\Psi^{-\infty}(X,E)$ is a Fréchet algebra. In particular, A is characterized by a smooth Schwartz kernel

$$k^A(x,y) \in C^\infty(X\times X, \pi_1^*(E)\otimes\pi_2^*(E)^*)$$

($\pi_1^*(E)\otimes\pi_2^*(E)^*$ is the vector bundle with fibre $\mathrm{Hom}(E_y,E_x)$ at $(x,y)\in X\times X$; §1.1.7, [**75**]). There is a (projectively) unique trace on $\Psi^{-\infty}(X,E)$, the *classical trace* (§4.3.2, [**75**]):

$$\mathrm{Tr}:\Psi^{-\infty}(X,E)\to\mathbb{C}, \qquad \mathrm{Tr}(A) := \int_X \mathsf{tr}\; k^A(x,x)v(x)dx. \tag{0.2.1}$$

where tr is the matrix trace on $\mathrm{End}(\mathbb{C}^N)$.

If X is an n-manifold with non-empty boundary Y, let $\widetilde{X}$ be a closed n-manifold such that $X \hookrightarrow \widetilde{X}$ smoothly and $E = \widetilde{E}_{|X}$ for a Hermitian vector bundle $\widetilde{E}\to\widetilde{X}$ of rank N, as in §0.1. Then the classical ψdos $\Psi(X,E)$ are defined from $\Psi(\widetilde{X},\widetilde{E})$ by truncation as follows. Let us consider the natural operators (§11, [**10**]):

- *restriction*: $r^+ : H^s(\widetilde{X},\widetilde{E}) \to H^s(X,E)$, $u\mapsto u_{|X}$, $\forall s\geq 0$,
- *extension by zero*: $e^+ : L^2(X,E)\to L^2(\widetilde{X},\widetilde{E})$,

$$e^+u(x) = \begin{cases} u(x) & \text{if } x\in X,\\ 0 & \text{if } x\in \widetilde{X}\setminus X;\end{cases}$$

Then r^+ and e^+ are mutually L^2-adjoint, i.e. for $u\in L^2(X,E)$ and $v\in L^2(\widetilde{X},\widetilde{E})$:

$$\langle e^+u,v\rangle_{\widetilde{X}} = \int_{\widetilde{X}} g_{\widetilde{x}}\left(e^+u,v\right)v(\widetilde{x})d\widetilde{x} = \int_X g_x\left(u,v_{|X}\right)v(x)dx + 0 = \langle u, r^+v\rangle_X.$$

Thus, $A\in\Psi^m(X,E)$ is defined as $A := r^+\widetilde{A}e^+$ for $\widetilde{A}\in\Psi^m(\widetilde{X},\widetilde{E})$. For A to be regular over the boundary (§1.2, [**26**]), we must assume a *transmission property*, i.e. in a collar neighbourhood of Y, for $(t,\tau)\in T^*\mathbb{R}$,

$$D^\beta_x D^\alpha_\xi \sigma^{\widetilde{A}}_{m-l}(0,y,-\tau,0) = e^{i\pi(m-l-|\alpha|)} D^\beta_x D^\alpha_\xi \sigma^{\widetilde{A}}_{m-l}(0,y,\tau,0). \tag{0.2.2}$$

Then $A: H^s(X,E)\to H^{s-m}(X,E)$, $s>0$, will be continuous. In particular, if A is considered together with a boundary operator $T: C^\infty(X,E)\to C^\infty(Y,E')$, i.e.

a *trace* (§1.2, [**26**]), a *singular Green* operator G, a *Poisson* operator K, and a ψdo over the boundary S, then the matrix:

$$\begin{pmatrix} A+G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(X,E) \\ \oplus \\ C^\infty(Y,E') \end{array} \rightarrow \begin{array}{c} C^\infty(X,F) \\ \oplus \\ C^\infty(Y,F') \end{array}$$

belongs to the *Boutet de Monvel calculus* or calculus of *Pseudodifferential boundary operators* (ψdbo). These operators in fact form an algebra which encompasses the calculus of elliptic differential boundary problems and their solution operators. Since we will not work with such algebra in general, we will not report the details of each of the aforementioned operators. We only want to remark that such an algebra can be seen as a good extension of the algerba of classical ψdos on closed manifolds, at least with respect to the *residue trace* (see Chapter 6), which in fact is generalized to the Boutet de Monvel algebra and is the unique trace there. We refer to [**26**] for further details on Boutet de Monvel calculus.

0.3. Decomposition of differential forms near the boundary

The main references in this section will be [**23**] and [**68**]. We will denote the vector bundle of differential forms on X of degree k by $\Lambda^k(X) := \Lambda^k(T^*X)$, and the space of smooth k-forms by $\Omega^k(X) := C^\infty(X, \Lambda^k(X))$, $k \in \{0, \dots, n = \dim X\}$. We recall that $\Lambda^k(X)$, and hence $\Omega^k(X)$, are both graded, i.e. $\Lambda(X) := \bigoplus_{k=0}^n \Lambda^k(X)$ and $\Omega(X) := \bigoplus_{k=0}^n \Omega^k(X)$. Together with the exterior derivative $d_k := d_{|\Omega^k(X)}$, $(\Omega^k(X), d_k)$ will be called *de Rham complex.*

Let $H^k(X,\mathbb{C})$ denote de Rham cohomology[2] and $*_k : \Omega^k(X) \to \Omega^{n-k}(X)$ the Hodge operator, arising from the metric on X. Since $*_{n-k}*_k = (-1)^{k(n-k)}$, then $*_k^{-1} = (-1)^{k(n-k)}*_{n-k}$. When X is closed, $*_k$ yields *Poincaré Duality*, i.e. $H^k(X,\mathbb{C}) \cong H^{n-k}(X,\mathbb{C})$ (§3.3, [**29**]). We also recall that $*_k$ turns $\Omega^k(X)$ into a Hilbert space via the inner product $\langle \alpha, \beta \rangle := \int_X \alpha \wedge *\beta$ and provides an adjoint for d_k, i.e. the *codifferential* $\delta_k : \Omega^{k+1}(X) \to \Omega^k(X)$,

$$\delta_k := (-1)^{n(k+1)+1} *_{n-k} d_{n-(k+1)} *_{k+1} .$$

The operator $d + \delta : \Omega(X) \to \Omega(X)$ is a first order differential operator, called *de Rham operator.*

[2]We have considered de Rham cohomology with complex coefficients, which is equivalent to de Rham cohomology with coefficients in any other field of characteristic zero by the *Universal Coefficient Theorem* (§3.1, [**29**]).

DEFINITION 0.3.1 (Definition 1.2.2, [**68**]). The *Hodge-Laplacian* (or *Laplace-de Rham operator*) is the map $\Delta : \Omega(X) \to \Omega(X)$ defined as $\Delta := d\delta + \delta d = (d+\delta)^2$. In particular, on smooth k-forms:

$$\Delta_k := \Delta_{|\Omega^k(X)} = d_{k-1}\delta_{k-1} + \delta_k d_k : \Omega^k(X) \to \Omega^k(X).$$

Both $d + \delta$ and Δ are self-adjoint elliptic differential operators on X closed.

When $Y = \partial X \neq \emptyset$, we write $\Omega^k(X)_{|Y} := C^\infty(Y, \Lambda^k(X)_{|Y})$ for the space of restrictions to the boundary $\omega_{|Y} = \gamma\omega$ of smooth k-forms $\omega \in \Omega^k(X)$. By product structure (0.1.1) on a collar neighbourhood $U \cong [0, c) \times Y$, we have the orthogonal decomposition:

$$\omega_{|U} = \omega_1 + dt \wedge \omega_2, \tag{0.3.1}$$

where $\omega_1 \in C^\infty([0, c)) \otimes \Omega^k(Y)$ and $\omega_2 \in C^\infty([0, c)) \otimes \Omega^{k-1}(Y)$, which corresponds to the decomposition $\Lambda(X)_{|Y} = \Lambda(Y) \oplus \Lambda(Y)$ into the ± 1-eigenspaces of the self-adjoint idempotent $\alpha(\gamma\omega) = \omega_1 - dt \wedge \omega_2$ (§4.1, [**23**]). In this way, we can define the following fundamental boundary operators:

DEFINITION 0.3.2 (§1.2, [**68**]). Consider the orthogonal decomposition of $\gamma\omega$ as in (0.3.1). Then the differential forms $\omega_1 \in \Omega^k(Y)$ and $\omega_2 \in \Omega^{k-1}(Y)$ are called *tangential* and *normal* components of $\gamma\omega$. Moreover, this decomposition defines the orthogonal (complementary) projections

$$\begin{aligned} \mathcal{R} : \Omega(X)_{|Y} &\to \Omega(Y) & \qquad \mathcal{A} : \Omega(X)_{|Y} &\to \Omega(Y) \\ \omega_{|Y} &\mapsto \omega_1 & \omega_{|Y} &\mapsto \omega_2. \end{aligned}$$

As in Example 0.1.1, the natural embedding $\iota : Y \hookrightarrow X$ defines by pull-back the surjection $\iota^* : \Omega(X) \to \Omega(Y)$. Since $\mathcal{R}\gamma = \iota^*$, the projection $\mathcal{R}$ does not depend on the metric g^X. On the other hand, $\mathcal{A}$ does, since it depends on a choice of normal tangent vector to the boundary (§4.1, [**23**], and §1.2, [**68**]).

PROPOSITION 0.3.3 (1.2.6, [**68**]). $\mathcal{R}$ and $\mathcal{A}$ are *Hodge adjoint* to each other, i.e. $*\mathcal{R} = \mathcal{A}*$. Moreover, $\mathcal{R}$ is d-invariant, while $\mathcal{A}$ is δ-invariant, i.e.:

$$\mathcal{R}(d\omega) = d(\mathcal{R}\omega) \quad \text{and} \quad \mathcal{A}(\delta\omega) = \delta(\mathcal{A}\omega).$$

REMARK 0.3.4. Because of Proposition 0.3.3, the complexes $(\Omega^k(X), d)$ and $(\Omega^k(X), \delta)$ can be refined to the complexes

$$d_k : \Omega^k_{\mathcal{R}}(X) \to \Omega^{k+1}_{\mathcal{R}}(X) \qquad \text{and} \qquad \delta_k : \Omega^{k+1}_{\mathcal{A}}(X) \to \Omega^k_{\mathcal{A}}(X),$$

where $\Omega^k_{\mathcal{R}}(X) = \{\omega \in \Omega^k(X) |\ \mathcal{R}\gamma\omega = 0\}$ and $\Omega^k_{\mathcal{A}}(X) = \{\omega \in \Omega^k(X) |\ \mathcal{A}\gamma\omega = 0\}$. In this case we say that $(\Omega^k_{\mathcal{R}}(X), d)$ (resp. $(\Omega^k_{\mathcal{A}}(X), \delta)$) corresponds to $(\Omega^k(X), d)$ with *relative* (resp. *absolute*) *boundary conditions* (§2.6, [**68**]).

REMARK 0.3.5. All the above can be generalize to smooth forms with coefficients in a *flat*[3] vector bundle $E \to X$ with rank N and connection ∇^E. In fact, all the previous definitions and results carry over to the *twisted* de Rham complex $(\Omega(X, E), d^E)$, where $\Omega(X, E) := C^\infty(X, \Lambda(X) \otimes E)$ and d^E is the *exterior covariant derivative* $d^E : \Omega(X, E) \to \Omega(X, E)$, defined as $d^E(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \otimes \nabla^E s$, for $\omega \in \Omega^k(X)$ and $s \in C^\infty(X, E)$.

0.4. Euler characteristic

Here, we recall the main definition and properties of the Euler characteristic. The main reference for simplicial homology and cohomology will be [**29**] and [**58**].

Let K be a simplicial complex, with subcomplex $L \subseteq K$. We denote their simplicial cohomology and relative cohomology groups of order k by $\mathcal{H}^k(K)$, $\mathcal{H}^k(L)$, and $\mathcal{H}^k(K, L)$, respectively. Then the *Euler characteristic* of K and L are the integers:

$$\chi(K) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(K) \quad \text{and} \quad \chi(L) = \sum_{k=0}^{n-1} (-1)^k \dim \mathcal{H}^k(L),$$

while $\chi(K, L) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(K, L)$ is the Euler characteristic of the pair (K, L), i.e. the *relative Euler characteristic* of the pair.

Let X be an n-manifold with boundary Y, possibly non-empty. It is well known that X admits a C^1-triangulation K, i.e. a simplicial complex, with a sub-triangulation L for Y (see [**59**]). Then, since simplicial homology is invariant under subdivision (Theorem 17.2, [**58**]), the Euler characteristic of X is invariantly defined as $\chi(X) := \chi(K)$. Analogously, $\chi(Y) := \chi(L)$, and $\chi(X, Y) := \chi(K, L)$ is the Euler characteristic of X relative to the boundary Y. They all are homotopy invariants of X, since they are defined at the level of cohomology.

There is a split short exact sequence $0 \to C(L) \to C(K) \to C(K, L) \to 0$ associated to the pair (K, L), which yields a long exact sequence (Theorem 43.1, [**58**]) $\cdots \to \mathcal{H}^{k-1}(L) \to \mathcal{H}^k(K, L) \to \mathcal{H}^k(K) \to \mathcal{H}^k(L) \to \cdots$ and the identity:

$$\chi(X) = \chi(X, Y) + \chi(Y). \tag{0.4.1}$$

[3] That is, $(\nabla^E)^2 = 0$, i.e. the curvature tensor vanishes.

The Euler characteristic of a *closed* manifold can also be expressed in terms of its de Rham cohomology. In fact, by the *de Rham Theorem* (Chapter 5, [**84**]), $\mathcal{H}^k(X) \cong H^k(X, \mathbb{C})$; thus $\chi(X) = \sum_{k=0}^{n}(-1)^k \dim H^k(X, \mathbb{C})$.

REMARK 0.4.1. By Poincaré Duality, $\chi(X) = 0$ if $n = \dim X$ is odd. Therefore (0.4.1) yields $\chi(X) = \chi(X, Y)$ when $n = \dim X$ is even.

If $Y \neq \emptyset$, then $\chi(X)$ and $\chi(X, Y)$ can be represented in terms of $H^k_{\mathcal{R}}(X, \mathbb{C})$ and $H^k_{\mathcal{A}}(X, \mathbb{C})$, i.e. the cohomology of $(\Omega^k_{\mathcal{R}}(X), d)$ and $(\Omega^k_{\mathcal{A}}(X), \delta)$, respectively, with complex coefficients. In fact, from §4.1, [**23**], we know that

$$\mathcal{H}^k(X, Y) \cong H^k_{\mathcal{R}}(X, \mathbb{C}) \qquad \text{and} \qquad \mathcal{H}^k(X) \cong H^k_{\mathcal{A}}(X, \mathbb{C}).$$

Thence, $\chi(X) = \sum_{k=0}^{n}(-1)^k \dim\, H^k_{\mathcal{A}}(X, \mathbb{C})$ and $\chi(X, Y) = \sum_{k=0}^{n}(-1)^k \dim H^k_{\mathcal{R}}(X, \mathbb{C})$.

The Hodge operator $*$ induces Poincaré Duality for manifolds with boundary, $H^k_{\mathcal{R}}(X, \mathbb{C}) \cong H^{n-k}_{\mathcal{A}}(X, \mathbb{C})$ (Corollary 2.6.2, [**68**]), which yields $\chi(X) = (-1)^n \chi(X, Y)$. Hence, in conclusion:

LEMMA 0.4.2 (4.1.5, [**23**]).

$$\chi(X) = \begin{cases} \chi(X, Y) & \text{if } n \text{ even}, \\ -\chi(X, Y) = \frac{1}{2}\chi(Y) & \text{if } n \text{ odd}. \end{cases} \tag{0.4.2}$$

Finally, *Mayer-Vietoris Theorem* (§3.1, [**29**]) provides a quasi-additive formula when two manifolds are glued along diffeomorphic components of their boundaries, i.e. if X_i, $i = 1, 2$, is an n-dimensional manifold with connected component of the boundary $Y_i \subseteq \partial X_i$ and $Y_1 \xrightarrow{\phi} Y_2$, then $\chi(X_1 \cup_\phi X_2) = \chi(X_1) + \chi(X_2) - \chi(Y_1)$, which translates into:

$$\chi(X_1 \cup_\phi X_2) = \chi(X_1) + \chi(X_2) \quad \text{if } n = \dim X_i \text{ is even}. \tag{0.4.3}$$

Notice that $\chi(Y_1) = \chi(Y_2)$ and $\chi(X_1 \cup_\phi X_2)$ does not depend on ϕ, i.e. it is *cut-and-paste invariant* (Chapter 1, [**36**]). Also, Lemma 0.4.2 yields, for $X := X_1 \cup_\phi X_2$,

$$\chi(X, \partial X) = \chi(X_1, \partial X_1) + \chi(X_2, \partial X_2) + \chi(Y_1) \quad \text{and}$$

$$\chi(X, \partial X) = \chi(X_1, \partial X_1) + \chi(X_2, \partial X_2) \quad \text{if } n = \dim X_i \text{ is even}. \tag{0.4.4}$$

REMARK 0.4.3. In the context of Remark 0.3.5, k-forms have coefficients in the fibre of a flat vector bundle E. Thus, de Rham Theorem generalizes to this context, yielding $\mathcal{H}^k(X, Y, E) \cong H^k_{\mathcal{R}}(X, E)$ and $\mathcal{H}^k(X, E) \cong H^k_{\mathcal{A}}(X, E)$. Now, if C be a chain complex of free abelian groups with homology groups $\mathcal{H}_k(C)$, then the cohomology groups $\mathcal{H}^k(C;G)$ of the cochain complex $\mathrm{Hom}(\mathcal{H}_k(C),G)$ satisfy the split exact sequence:

$$0 \longrightarrow \mathrm{Ext}(\mathcal{H}_{k-1}(C),G) \longrightarrow \mathcal{H}^k(C;G) \longrightarrow \mathrm{Hom}(\mathcal{H}_k(C),G) \longrightarrow 0,$$

(Universal Coefficient Theorem; 3.2, [**29**]). Then, since $G = E$ is (locally) a vector space and $\mathcal{H}_k(C) = \mathcal{H}_k(X)$ are free groups, $\mathrm{Ext}(\mathcal{H}_k(X),E) = 0$ (see [**29**]) and $\mathcal{H}^k(X,E) \cong \mathrm{Hom}(\mathcal{H}_k(X),E)$. Therefore, $\dim \mathcal{H}^k(X,E) = \dim \mathcal{H}_k(X) \cdot \dim E$.

Therefore, if we define:

$$\chi(X,E) = \sum_{k=0}^{n} (-1)^k \dim \ H^k_{\mathcal{A}}(X,E), \quad \chi(X,Y,E) = \sum_{k=0}^{n} (-1)^k \dim H^k_{\mathcal{R}}(X,E),$$

$$\text{and } \chi(Y,E) = \sum_{k=0}^{n-1} (-1)^k \dim H^k(Y,E),$$

we can conclude $\chi(X,E) = \chi(X) \cdot \mathrm{rk}(E)$ and similarly for $\chi(X,Y,E)$ and $\chi(Y,E)$.

Part 1

Logarithmic TQFT

CHAPTER 1

Logarithmic structures and LogTQFT

In this chapter we recall the definitions of logarithmic representations, traces and determinants and provide some classical examples of logarithms (the local and global logarithm on $\mathrm{GL}(n,\mathbb{C})$ and the Fredholm index). As part of this introduction, we prove some general equivalent conditions for the uniqueness of log, trace and det.

Then, we will present the main object of this work: the logarithmic representation of a symmetric monoidal categoy $\mathbf{C}$, or *log-functor* (§1.4), which is called *LogTQFT* if $\mathbf{C} = \mathbf{Cob}_n$, the catory of n-dimensional cobordisms. This categorical construction requires some preparation, which we summarize from [**72**], where log-functors appeared for the first time.

At the end of the chapter we state and prove a new result for 2-dimensional unoriented LogTQFTs, which classifies them in terms of the logarithmic representation of the unit disc and the Euler characteristic of the cobordisms.

1.1. Logarithms and log-determinants structures

The following definitions can be found in §4.1, [**73**].

Definition 1.1.1. Let $\mathcal{S}$ be a topological semigroup and $\mathcal{T}$ a unital locally convex topological algebra. Then a (global) *logarithmic representation* (or simply *logarithm*) of $\mathcal{S}$ is a homomorphism:

$$\log : \mathcal{S} \to \frac{(\mathcal{T},+)}{[\mathcal{T},\mathcal{T}]}, \qquad a \mapsto \log a$$

satisfying, for every $a, b \in \mathcal{S}$, a log-additive property $\log ab = \log a + \log b$, meaning:

$$\log ab - \log a - \log b = \sum_{i=1}^{N} [c_i, c_i'] \in [\mathcal{T},\mathcal{T}] \quad \text{for some } c_i, c_i' \in \mathcal{T}, \tag{1.1.1}$$

where $[c_i, c_i'] := c_i c_i' - c_i' c_i$ is the *commutator* of c_i and c_i' and $[\mathcal{T},\mathcal{T}]$ is the subgroup of $(\mathcal{T},+)$ of finite sums of commutators.

Remark 1.1.2. A logarithm is *local* if, for each $a \in \mathcal{S}$, it is only defined for an open neighbourhood $\mathcal{U}$ of a, i.e. $\log_{\mathcal{U}} : \mathcal{U} \to \mathcal{T}$ such that for any $a, b \in \mathcal{S}$ there exist neighbourhoods $\mathcal{U}, \mathcal{V}, \mathcal{W}$ such that $\log_{\mathcal{W}} ab - \log_{\mathcal{U}} a - \log_{\mathcal{V}} b \in [\mathcal{T}, \mathcal{T}]$. In this case, $\log_{\mathcal{U}}$ is called a *branch* of the log. An example will be given in §1.3.1.

Thus, there exists an abelian group of logarithm representations of $\mathcal{S}$ into $\mathcal{T}$,

$$\mathbb{L}\mathrm{og}(\mathcal{S}, \mathcal{T}) := \mathrm{Hom}\left(\mathcal{S}, \frac{(\mathcal{T}, +)}{[\mathcal{T}, \mathcal{T}]}\right).$$

REMARK 1.1.3. By (1.1.1), if $p \in \mathcal{S}$ is idempotent, i.e. $p^2 = p$, then $\log p = 0$. In particular, if $\mathcal{S}$ is a monoid with unit ι, then $\log \iota = 0$. All other standard properties of the logarithm naturally follow ([**73**]).

DEFINITION 1.1.4. A homomorphism of groups $\tau : (\mathcal{T}, +) \to (\mathcal{U}, +)$ is said to be a *trace* on $\mathcal{T}$ if it vanishes on commutators: $\tau([c, \tilde{c}]) = 0$, i.e. $[\mathcal{T}, \mathcal{T}] \subset \ker(\tau)$. The abelian group of traces is denoted by

$$\mathbb{T}\mathrm{race}(\mathcal{T}, \mathcal{U}) := \mathrm{Hom}(\mathcal{T}/[\mathcal{T}, \mathcal{T}], \mathcal{U}).$$

DEFINITION 1.1.5. A *log-determinant* (or *log-character*, or *τ-character*) is the composition $\tau \circ \log : \mathcal{S} \to \mathcal{U}$ of a logarithmic representation of $\mathcal{S}$ with a trace τ.

By the linearity of τ and (1.1.1) we have the *additive property of log-characters*:

$$\tau(\log ab) = \tau(\log a) + \tau(\log b) \ \forall a, b \in \mathcal{S}.$$

DEFINITION 1.1.6. If $e : (\mathcal{U}, +, \cdot) \to (\mathcal{V}, +, \cdot)$ is an exponential map, i.e. a homomorphism of unital rings such that $e(a + b) = e(a) \cdot e(b)$, then the triple $(\log, \tau, e)$ is called a *determinant structure* and a *determinant* associated to the triple is defined by the composition

$$\det_{\tau, e} := e \circ \tau \circ \log .$$

It follows that a determinant functional has a natural multiplicative property:

$$\det\, ab = \det\, a \cdot \det\, b \quad \forall a, b \in \mathcal{S}.$$

1.2. Uniqueness of logarithm, trace and determinant

Here, we present the proofs of three similar lemmas about equivalent conditions for the uniqueness of logarithm, trace and determinant. The main technical result we need is the *Snake's Lemma*:

THEOREM 1.2.1 (§VIII.4, [**50**]). In an abelian category[1], let us consider the following morphism of short exact sequences, i.e. the triple of morphisms (f, g, h) such that the following diagram commutes:

[1] Such as **Ab**, the category of abelian groups and group homomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"] & B \arrow[r, "e"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0.
\end{tikzcd}$$

Then there is a morphism $\delta : \ker(h) \to \operatorname{coker}(f)$ such that the following sequence is exact:

(1.2.1) $$0 \to \ker(f) \to \ker(g) \to \ker(h) \stackrel{\delta}{\to} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

REMARK 1.2.2. Notation here will try to be consistent with the common use of additive notation for abelian groups and multiplicative notation for non-abelian groups. Thus the unit elements will be respectively denoted by 0_B (or just 0) when $(B, +)$ is abelian and 1_G (or just 1) for $(G, \cdot)$ non-abelian.

REMARK 1.2.3. From now on, let R be a commutative unital ring and denote by R^* the subring of units of R. Notice that a trace on an R-module is in particular an R-linear homomorphism.

LEMMA 1.2.4 (Uniqueness of trace). Let B be a ring and an R-module and let $\tau : (B, +) \to R$ be a trace. Consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {[B,B]} \arrow[r, "i"] \arrow[d, "i"] & B \arrow[r, "\sigma"] \arrow[d, "id"] & \frac{B}{[B,B]} \arrow[r] \arrow[d, "\pi_\tau"] & 0 \\
0 \arrow[r] & \ker(\tau) \arrow[r, "i"] & B \arrow[r, "\tau"] & R \arrow[r] & 0.
\end{tikzcd}$$

Then the following are equivalent:

(1) $\frac{B}{[B,B]} \stackrel{\pi_\tau}{\to} (R, +)$ is an isomorphism of abelian groups;
(2) $\ker(\tau) = [B, B]$;
(3) if $\xi \in B$ with $\tau(\xi) \in R^*$, then $\forall \beta \in B$ we can write:

$$\beta = \tau(\beta)\tau(\xi)^{-1}\xi + \sum_{j=1}^{n} [\delta_j, \delta'_j]$$

for some $\delta_j, \delta'_j \in B$ depending on β and ξ;
(4) τ is projectively unique, i.e. for any other trace $\widetilde{\tau} : B \to R$ there exist $r \in R$ such that $\widetilde{\tau} = \tau \cdot r$;
(5) $\mathbb{T}\mathrm{race}(B, R) \cong (R, +)$.

LEMMA 1.2.5 (Uniqueness of logarithm). Let G be a group and consider its commutator subgroup $G' = \{ghg^{-1}h^{-1} \mid g, h \in G\}$. For B an R-module, let $\log : G \to B$ be a logarithm and consider the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & G' \arrow[r, "i"] \arrow[d, "i"] & G \arrow[r, "\pi"] \arrow[d, "id"] & \frac{G}{G'} \arrow[r] \arrow[d, "\pi_{\log}"] & 1 \\
1 \arrow[r] & \ker(\log) \arrow[r, "i"] & G \arrow[r, "\log"] & \frac{B}{[B,B]} \arrow[r] & 0.
\end{tikzcd}$$

Then the following are equivalent:

(1) $\frac{G}{G'} \overset{\pi_{\log}}{\to} \frac{B}{[B,B]}$ is an isomorphism of abelian groups and the short exact sequence $1 \to G' \to G \to \frac{G}{G'} \to 1$ is split $J' : \frac{G}{G'} \to G$;

(2) $\ker(\log) = G'$ and $1 \to \ker(\log) \to G \overset{\log}{\to} \frac{B}{[B,B]} \to 1$ is a split short exact sequence $J : \frac{B}{[B,B]} \to G$;

(3) for a given splitting $J : \frac{B}{[B,B]} \to G$ of log, any $g \in G$ can be written:

$$g = \Pi_k \{l_k, l'_k\} \cdot J(\log g)$$

for some $l_k, l'_k \in G$ depending on g and J, where $\{l_1, l_2\} = l_1 l_2 l_1^{-1} l_2^{-1}$.

Then log is the unique logarithm split by J.

Lemma 1.2.6 (Uniqueness of determinant). Let G be a group and G' its commutator subgroup as in Lemma 1.2.5. Let $\det : G \to R^*$ be a determinant and consider the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & G' \arrow[r, "i"] \arrow[d, "i"] & G \arrow[r, "\pi"] \arrow[d, "id"] & \frac{G}{G'} \arrow[r] \arrow[d, "\pi_{\det}"] & 1 \\
1 \arrow[r] & \ker(\det) \arrow[r, "i"] & G \arrow[r, "\det"] & R^* \arrow[r] & 1.
\end{tikzcd}$$

Then the following are equivalent:

(1) $\frac{G}{G'} \overset{\pi_{\det}}{\to} \frac{B}{[B,B]}$ is an isomorphism of abelian groups and the short exact sequence $1 \to G' \to G \to \frac{G}{G'} \to 1$ is split $j' : \frac{G}{G'} \to G$;

(2) $\ker(\det) = G'$ and the short exact sequence $1 \to \ker(\det) \to G \overset{\det}{\to} R^* \to 1$ is split $j : R^* \to G$;

(3) for a splitting $j : R^* \to G$ of det any $g \in G$ can be written:

$$g = \Pi_k \{h_k, h'_k\} \cdot j(\det g)$$

for some $h_k, h'_k \in G$ depending on g and j.

Then det is the unique determinant split by j.

Remark 1.2.7. A priori, the homomorphisms of groups log, τ and det are not required to be surjective, hence the second row of the diagrams need not be exact.

But the R-linearity hypothesis for τ and the split hypothesis for log and det, or assuming $\pi_{\log}$ and $\pi_{\det}$ invertible, will provide exactedness for the second row.

Moreover, we must notice that, except for the case of the trace, our commutative diagrams belong to **Grp**, the category of groups and homomorphisms of groups, which is not abelian or even additive. However, the morphisms involved are the inclusions i and identity id, for which kernels and cokernels are defined and trivial:

$$\ker(i : G' \to \ker\log) = \ker(id : G \to G) = \operatorname{coker}(id) = \{1\},$$

(likewise for the determinant), and the subgroup G' is normal in G. Therefore, $\delta : \ker(h) \to \operatorname{coker}(f)$ of Theorem 1.2.1 exists and is well-defined also in these cases. In fact, let us consider log (for det the proof works in the same way) and let $z \in \ker\pi_{\log}$; since π is surjective, $\exists\ y \in G$ such that $\pi(y) = z$ (specifically, $z = yG'$). The identity pushes down y to itself and since the diagram commute, i.e. $\log\circ id = \pi_{\log}\circ\pi$, we have $\log(y) = \pi_{\log}\circ\pi(y) = \pi_{\log}(z) = 1$, so $y \in \ker\log \leq G$. Let $\pi_{G'} : \ker\log \to \operatorname{coker}(i)$, with $i : G' \to \ker\log$. By definition of δ, $\delta z = \pi_{G'}\circ i^{-1}\circ id\circ\pi^{-1}(z) = \pi_{G'}(y) = yG'$ and if $x \in \pi^{-1}(z)$, i.e. $xG' = z = yG'$, we obtain $\delta xG' = xG'$. Hence δ is independent of the choice of representative of yG' and is the identity, and (1.2.1) is exact if and only if $\pi_{\log}$ is an isomorphism.

Proof of Lemma 1.2.4. Clearly, $\ker(i) = \ker(id) = \operatorname{coker}(id) = \{0\}$, where $i : [B,B] \to \ker\tau$. Moreover, τ is surjective because R-linear (see Remark 1.2.8), hence π_τ is surjective as well (by commutativity of the diagram) and $\operatorname{coker}(\pi_\tau) = \{0\}$. Since the category of R-modules is abelian, Theorem 1.2.1 applies and

$$0 \to 0 \to 0 \to \ker(\pi_\tau) \xrightarrow{\delta} \operatorname{coker}(i) \to 0 \to 0 \to 0$$

is exact, where $i : [B,B] \to \ker(\tau)$ and $\operatorname{coker}(i) := \ker(\tau)/[B,B]$. Hence δ is an isomorphism.

(1) $\Leftrightarrow$ (2) π_τ isomorphism $\Leftrightarrow \ker(\pi_\tau) = \{0\} \Leftrightarrow \operatorname{coker}(i) \overset{\delta}{\cong} \{0\} \Leftrightarrow \ker(\tau) = [B,B]$.

(2) $\Rightarrow$ (3) If $\ker(\tau) = [B,B]$ and $\xi \in B$ such that $\tau(\xi) \in R^*$, then $\forall\beta \in B$ $\beta - \tau(\beta)\tau(\xi)^{-1}\xi \in \ker(\tau)$. Hence there exist $\delta_j, \delta_j' \in B$, $1 \leq j \leq n$, such that $\beta - \tau(\beta)\tau(\xi)^{-1}\xi = \sum_{j=1}^n [\delta_j, \delta_j']$.

(3) $\Rightarrow$ (4) If $\widetilde{\tau} : B \to R$ is another trace, then $[B,B] \subseteq \ker(\widetilde{\tau})$ and therefore $\widetilde{\tau}(\beta) = \widetilde{\tau}\left(\tau(\beta)\tau(\xi)^{-1}\xi + \sum_{j=1}^n [\delta_j, \delta_j']\right) = \tau(\beta)\tau(\xi)^{-1}\widetilde{\tau}(\xi)$. Hence $\widetilde{\tau} = \tau\cdot r$ with $r = \tau(\xi)^{-1}\widetilde{\tau}(\xi)$.

(4) $\Rightarrow$ (5) Since for any other trace $\widetilde{\tau}$ we have $\widetilde{\tau} = \tau \cdot r$, with $r = \tau(\xi)^{-1}\widetilde{\tau}(\xi)$, this defines a homomorphism $\widetilde{\tau} \to r$ which is clearly one-to-one and onto. Hence $\mathbb{T}\mathrm{race}(B, R) \cong (R, +)$.

(5) $\Rightarrow$ (2) Since $\mathbb{T}\mathrm{race}(B, R) \cong (R, +)$, then $\dim \mathrm{Hom}(B/[B, B], R) = 1$. As $\mathrm{Hom}(B/[B, B], R)$ is the dual of $B/[B, B]$, then $\dim B/[B, B] = 1$. Therefore $B/[B, B] \cong (R, +)$. Let $\mathbf{t}$ be the generator of $\mathbb{T}\mathrm{race}(B, R)$. Then $\forall \widetilde{\tau} \in \mathbb{T}\mathrm{race}(B, R)$ $\exists\ s \in R$ such that $\widetilde{\tau} = \mathbf{t} \cdot s$, therefore $\ker(\widetilde{\tau}) = \ker(\mathbf{t})$. Let us suppose that $[B, B]$ is a proper subgroup of $\ker(\mathbf{t})$. Then $B/\ker(\mathbf{t})$ is a proper subgroup of $B/[B, B]$. Therefore, since $B/\ker(\mathbf{t})$ cannot be trivial, as $\mathbf{t}$ is not, it must be 1-dimensional as well, and therefore $\ker(\mathbf{t}) = [B, B]$.

$\square$

Proof of Lemma 1.2.5. (1) $\Rightarrow$ (3) Since $\pi_{\log}$ is surjective, so is $\log = \pi_{\log} \circ \pi$ and the second row is exact. Since the first row is right split $J' : G/G' \to G$, i.e. $\pi \circ J' = id_{G/G'}$, then the second row is right split as well, $J : B/[B, B] \to G$. In fact, if we define $J := J' \circ \pi_{\log}^{-1}$, then:

$$\log \circ J = \pi_{\log} \circ \pi \circ J' \circ \pi_{\log}^{-1} = \pi_{\log} \circ\ id_{G/G'} \circ \pi_{\log}^{-1} =\ id_{B/[B,B]}.$$

Since the first row is exact, we can write $G = G' \cdot J'(G/G') = G' \cdot J(\pi_{\log}(G/G'))$, i.e. each $g \in G$ can be written as the product of an element $g' \in G'$, which is a finite product of commutators, and one $h \in J'(G/G') = J \circ \pi_{\log}(G/G') = J(B/[B, B])$:

$$g = \Pi_k\{h_k, h'_k\} \cdot J' \circ \pi(h) = \Pi_k\{h_k, h'_k\} \cdot J \circ \pi_{\log} \circ \pi(h) = \Pi_k\{h_k, h'_k\} \cdot J(\log h).$$

Clearly $\log g = \log h$, hence statement 3 holds.

(3) $\Rightarrow$ (2) If the second row is split, then log is surjective and J injective and the sequence is exact. As $B/[B, B]$ is abelian, its unit is denoted 0, while the unit of $(G, \cdot)$ is denoted 1 (See Remark 1.2.2). Hence, if $g \in \ker \log$ then:

$$g = \Pi_k\{l_k, l'_k\} \cdot J(\log g) = \Pi_k\{l_k, l'_k\} \cdot J(0) = \Pi_k\{l_k, l'_k\} \cdot 1 = \Pi_k\{l_k, l'_k\} \in G'.$$

(2) $\Rightarrow$ (1) By definition of split, $\log \circ J = id_{B/[B,B]}$, hence log is surjective and the row is exact. From Remark 1.2.7, $\delta : \ker \pi_{\log} \to \ker \log / G'$ is the identity, so if $\ker \log = G'$, then $\ker(\pi_{\log}) = \{1\}$. Also, since $\log = \pi_{\log} \circ \pi$ and log is surjective, then $\pi_{\log}$ surjective, too, i.e. $\pi_{\log}$ is an isomorphism. Finally, if $J : B/[B, B] \to G$ is a right split for the second row, then $J' := J \circ \pi_{\log}$ defines a right split for the first row.

We now show that there is a unique log split by J if one of this equivalent conditions is satisfied. Let $\widetilde{\log}$ be another logarithm split by J, i.e. $\widetilde{\log} \circ J = id_{B/[B,B]}$. Hence $\widetilde{\log}$ vanishes on products of commutators, thus from (3) we have that $\forall g \in G$ $\widetilde{\log}(g) = \widetilde{\log}(\Pi_k\{l_k, l'_k\} \cdot J(\log g)) = \widetilde{\log} \circ J \circ \log g = \log g$, hence uniqueness.

□

PROOF OF LEMMA 1.2.6. The proof is very similar to the previous one for log, so we will give a brief sketch.

(1) $\Rightarrow$ (3) If π_{det} is surjective, so is $\det = \pi_{\text{det}} \circ \pi$, hence the second row is exact. If the first row is also right split $j' : G/G' \to G$, then so is the second row via $j := j' \circ \pi_{\text{det}}^{-1} : R^* \to G$. Since the first row is exact, we can write $G = G' \cdot j'(G/G') = G' \cdot j(\pi_{\text{det}}(G/G'))$ and every $g \in G$ can be written as the product of an element $g' \in G'$ and $h \in j'(G/G') = j \circ \pi_{\text{det}}(G/G') = j(R^*)$, which yields $g = \Pi_k\{h_k, h'_k\} \cdot j(\det g)$.

(3) $\Rightarrow$ (2) If the second row is split, then det is surjective, j is injective, and the sequence is exact. By the decomposition of $g \in G$ we have that if $g \in \ker \det$ then $g = \Pi_k\{h_k, h'_k\} \cdot j(\det g) = \Pi_k\{h_k, h'_k\} \cdot j(1_{R^*}) = \Pi_k\{h_k, h'_k\} \cdot 1_G \in G'$.

(2) $\Rightarrow$ (1) As for (1) $\Rightarrow$ (3), the splitting of the second row yields the splitting of the first, and since det is surjective, so is π_{det}. Also, from Remark 1.2.7, $\delta : \ker(\pi_{\text{det}}) \to \ker(\det)/G'$ is the identity, thus $\ker(\det) = G'$ yields $\ker(\pi_{\text{det}}) = \{1\}$.

Hence uniqueness follows by the same argument used for the logarithm.

□

REMARK 1.2.8. The hypothesis of τ R-linear assures that τ is surjective: in fact, for $\xi \in B$ such that $\tau(\xi) \in R^*$, $\tau(\alpha\tau(\xi)^{-1} \cdot \xi) = \alpha\tau(\xi)^{-1}\tau(\xi) = \alpha$ $\forall \alpha \in R$. Thus the corresponding sequences is exact. It also assures the lower sequence to split: in fact, we can define $K : R \to B$ with $K(r) = r\tau(\xi)^{-1}\xi$, which is injective, such that $\tau \circ K = id_R$. As in the proof of Lemma 1.2.5, the composition $K \circ \pi_\tau$ makes the upper exact sequence split.

1.3. Some examples of log-structures

As two fundamental examples of log-structures and log-characters, we present the classical logarithm with its generalisation to a global logarithm on the universal cover of the Lie group $\mathrm{GL}(n, \mathbb{C})$ and the index of Fredholm operators on a separable Hilbert space.

1.3.1. The local and global logarithm on GL$(n,\mathbb{C})$. It is well-known that the complex logarithm is not holomorphic on $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$, but is so with respect to a *complex cut* $R_\theta = \{w \in \mathbb{C}|\ w = re^{i\theta},\ r \geq 0\}$, $\theta \in \mathbb{R}$, in which case it is called a *branch* $\log_\theta : \mathbb{C}\setminus R_\theta \to \mathbb{C}$. However, a *global* holomorphic logarithm can be defined on the universal cover of $\mathbb{C}^\times$, $\mathcal{U} := \{\gamma \mid \gamma : [0,1] \to \mathbb{C}^\times, \gamma(0) = 1\}/_\sim$ (where $\sim$ is the homotopy equivalence relative to $\{0,1\}$). In fact (§1.6.20, [**75**]), if we parametrize $\mathcal{U}$ as $(r,\varphi) \in (0,\infty)\times\mathbb{R}$, we can define $\log(r,\varphi) := \log r + i\varphi$ and log-additivity follows from the natural product on $\mathcal{U}$, inherited from $\mathbb{C}$. Such $\log : \mathcal{U} \to \mathbb{C}^\times$ is a *global section* of the (line) bundle associated to $\mathcal{U} \to \mathbb{C}^\times$ via the representation $\rho : \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \to \mathrm{End}\mathbb{C} \cong \mathbb{C}$, $\rho(m)(\lambda) = \lambda - i2\pi m$:

$$\mathcal{L} := \mathcal{U}\times_{\mathbb{Z}}\mathbb{C} = \{[(|z|,\varphi),\lambda] \mid ((|z|,\varphi),\lambda) \sim ((|z|,\varphi)\cdot m, \rho(m)^{-1}\lambda),\ m \in \mathbb{Z}\}.$$

In other words, log is a holomorphic $\mathbb{Z}$-equivariant function on $\mathcal{U}$, i.e.

$$\log((r,\varphi)\cdot m) = \rho(m)^{-1}(\log(r,\varphi)),$$

with $(r,\varphi)\cdot m = (r,\varphi+2\pi m)$ the natural right action of $\mathbb{Z}$ on $\mathcal{U}$; the branches $\log_\theta$ are, instead, local sections of $\mathcal{U}$.

In a similar way, holomorphic functional calculus can define a logarithm for GL$(n,\mathbb{C})$ only locally, i.e. as a branch:

$$\log_\theta A := \frac{i}{2\pi}\int_{\mathscr{C}_\theta} \log_\theta\lambda\ (A-\lambda I)^{-1}d\lambda, \tag{1.3.1}$$

for an annulus $\mathscr{C}_\theta$ centred at 0, enclosing spec(A) and cut by R_θ (i.e. a *Laurent loop* as in §6.2.2). In fact, (1.3.1) is local as it defines a map $\log_\theta : U_\theta \to \mathrm{End}\mathbb{C}^n$, where

$$U_\theta = \{C \in \mathrm{GL}(n,\mathbb{C}) | \exists\epsilon > 0 \text{ s.t. } \|A - C\| < \epsilon,\ \mathrm{spec}(C) \subset \mathbb{C}\setminus R_\theta\}.$$

The branches satisfy

$$\log_{\theta+2\pi m} A = \log_\theta A + i2\pi m I, \tag{1.3.2}$$

and $\log_\theta A = \log_\varphi A + i2\pi\ \Pi_{\theta,\varphi}$ if $|\theta-\varphi| < 2\pi$, where $\Pi_{\theta,\varphi} = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}}(A-\lambda I)^{-1}\ d\lambda$ is the projection onto the direct sum of the eigenspaces of A corresponding to those eigenvalues inside the contour $\Gamma_{\theta,\varphi}$, which is the portion of annulus enclosing spec(A) cut by R_θ and R_φ. The log-additivity is a consequence of the *Campbell-Hausdorff formula* (§2.4, [**75**]):

$$\log_\theta AB - \log_\varphi A - \log_\phi B \in [\mathrm{End}(\mathbb{C}^n), \mathrm{End}(\mathbb{C}^n)] + P(\mathrm{End}(\mathbb{C}^n)),$$

with $P(\mathrm{End}(\mathbb{C}^n))$ the vector space of finite sums of projectors in $\mathrm{End}(\mathbb{C}^n)$.

Since $\pi_1(\mathrm{GL}(n,\mathbb{C})) \cong \mathbb{Z}$ as in the one dimensional case, we can once again consider the universal cover:

$$\mathcal{U}_n := \{\gamma \mid \gamma : [0,1] \to \mathrm{GL}(n,\mathbb{C}), \gamma(0) = I\}/_\sim,$$

a principal $\mathbb{Z}$-bundle over $\mathrm{GL}(n,\mathbb{C})$, and identify an element $[\gamma] \in \mathcal{U}_n$, $\gamma(1) = A$, with the pair (A,φ). With the representation $\rho : \mathbb{Z} \to \mathrm{End}(\mathrm{End}(\mathbb{C}^n))$ defined as $\rho(m)(A) = A - i2\pi m I$, we can then form the associated vector bundle:

$$\mathcal{V}_\rho := \mathcal{U}_n \times_\rho \mathrm{End}(\mathbb{C}^n) = \{[(A,\theta), L] \mid ((A,\varphi), L) \sim ((A,\varphi)\cdot m, \rho(m)^{-1}L),\ m \in \mathbb{Z}\},$$

where $(A,\varphi)\cdot m = (A, \varphi + 2\pi m)$. Then $\log_\theta A \in \mathrm{End}(\mathbb{C}^n)$ is a local section of $\mathcal{U}_n \to GL(n,\mathbb{C})$, while a global logarithm $\log : \mathcal{U}_n \to \mathrm{End}(\mathbb{C}^n)$ can be defined as

$$\log(A,\varphi) := \int_0^1 \gamma(t)^{-1} d\gamma(t), \qquad [\gamma] = (A,\varphi).$$

In fact, locally, $\int_0^1 \gamma(t)^{-1} d\gamma(t) = \int_0^1 d\log_\theta \gamma(t) = \log_\theta A$, and by (1.3.2) we obtain the $\mathbb{Z}$-equivariance of such log:

$$\log((A,\theta)\cdot m) = \log(A, \theta + 2\pi m) = \log_{\theta+2\pi m} A = \log_\theta A + i2\pi m I = \rho(m)^{-1}(\log(A,\theta)).$$

1.3.2. The index of Fredholm operators. The algebra $\mathcal{B}(H)$ of bounded linear operators on a separable Hilbert space H has a unique trace if and only if $\dim H < \infty$, and has no trace when $\dim H = \infty$ (§1.3, [**75**]). However, $\mathcal{B}(H)$ contains a tower of proper ideals that admit traces, the *Schatten ideals*:

$$\mathcal{F}(H) := \mathcal{C}_0 \subset \cdots \subset \mathcal{C}_p \subset \cdots \subset \mathcal{C}_\infty := \mathcal{C}(H),$$

where $\mathcal{F}(H) = \{A \in \mathcal{B}(H) | \dim \mathrm{Ran} A < \infty\}$ is the ideal of *finite rank operators*, while $\mathcal{C}(H)$, its closure in the norm topology, is the (maximal) ideal of compact operators. In particular, $\mathcal{F}(H)$ has a (unique) trace analogous to the classical trace on endomorphisms of finite-dimensional Hilbert spaces, still called *classical*, i.e. $\mathrm{Tr} A = \sum_{j=1}^\infty \langle Ae_j, e_j\rangle$, with $\{e_j\}_{j\in\mathbb{N}}$ any orthonormal basis of H (§1.3, [**75**]).

Definition 1.3.1 (From §2.2 and §2.8, [**75**]). $A \in \mathcal{B}(H)$ is a *Fredholm operator* if and only if there exists $P \in \mathcal{B}(H)$ such that $AP - I, PA - I \in \mathcal{F}(H)$. The space of Fredholm operators is a multiplicative semigroup denoted by $\mathrm{Fred}(H)$. Equivalently, $A \in \mathrm{Fred}(H)$ if and only if $\mathrm{ran}(A)$ and $\mathrm{ran}(A^*)$ are closed and $\dim\ker(A), \dim\ker(A^*) < \infty$. Clearly, if $A \in \mathrm{Fred}(H)$ then $P, A^* \in \mathrm{Fred}(H)$.

Definition 1.3.2. The *Fredholm index* of $A \in \mathrm{Fred}(H)$ is defined as:

$$\mathrm{ind}(A) := \dim\ker(A) - \dim\mathrm{coker}(A) = \dim\ker(A) - \dim\ker(A^*) \in \mathbb{Z}$$

It is well-known that the index is log-additive with respect to the composition of Fredholm operators, i.e. $\mathrm{ind}(AB) = \mathrm{ind}(A) + \mathrm{ind}(B)$. We can see that such log-additivity arises as a consequence of the log-additivity of a suitable logarithm defined on $\mathrm{Fred}(H)$.

First of all, a good candidate for a logarithm on $\mathrm{Fred}(H)$ is the commutator $[A, P]$, P a parametrix of $A \in \mathrm{Fred}(H)$. In fact, the dependence on P of $[A, P]$ lies in the commutator subgroup:

PROPOSITION 1.3.3 (§2.2.2, [**75**]). If $A \in \mathrm{Fred}(H)$ and P_1, P_2 two parametrices for A, then $[A, P] - [A, P'] \in [\mathcal{F}(H), \mathcal{F}(H)]$, i.e. $\pi([A, P]) = \pi([A, P'])$, where $\pi : \mathcal{F}(H) \to \mathcal{F}(H)/[\mathcal{F}(H), \mathcal{F}(H)]$ is the canonical projection.

Moreover, by the uniqueness of Tr and the first isomorphism theorem, there exists an isomorphism $\widetilde{\mathrm{Tr}} : \mathcal{F}(H)/[\mathcal{F}(H), \mathcal{F}(H)] \to \mathbb{C}$ such that $\mathrm{Tr} = \widetilde{\mathrm{Tr}} \circ \pi$. Therefore, for $\mathcal{F}_\pi(H) := \mathcal{F}(H)/[\mathcal{F}(H), \mathcal{F}(H)]$, we can define a logarithm $\log : \mathrm{Fred}(H) \to \mathcal{F}_\pi(H)$ as $\log A := \pi([A, P])$. In fact, it is proved in §2.2.2, [**75**], that it satisfies:

$$\log AB = \log A + \log B, \ \forall A, B \in \mathrm{Fred}(H),$$

and that $\mathrm{ind}(A) = \widetilde{\mathrm{Tr}}\,(\log A)$.

REMARK 1.3.4. The same considerations carry over to elliptic ψdos on a closed manifold X. If $A \in \Psi^m_{\mathrm{Ell}}(X, E)$, then (Theorem 19.2.3, [**32**]):

i) A is a Fredholm operator $H^s(X, E) \to H^{m-s}(X, E)$;
ii) $\ker(A) \subseteq C^\infty(X, E)$ (in particular, $\ker(A)$ is independent of s);
iii) $\mathrm{ran}(A) = \ker(A^*)^\perp$, with $A^* \in \Psi^m(X, E)$.

Thus, $\mathrm{ind}(A)$ is independent of s and there exists $P \in \Psi^{-m}_{\mathrm{Ell}}(X, E)$ such that $AP - I$ and $PA - I$ belong to $\Psi^{-\infty}(X, E)$. Hence, $[A, P]$ is independent of P and is trace class with respect to the classical trace Tr of smoothing ψdos (0.2.1). From §2.8, [**75**], P can be chosen in such a way that $AP - I = -P_{\ker(A^*)}$ and $PA - I = -P_{\ker(A)}$ and a logarithm is defined as:

$$\log : \Psi_{\mathrm{Ell}}(X, E) \to \left(\frac{\Psi^{-\infty}(X, E)}{[\Psi^{-\infty}(X, E), \Psi^{-\infty}(X, E)]}, \widetilde{\mathrm{Tr}} \right), \quad A \mapsto \pi\left([A, P]\right).$$

Hence, $\mathrm{ind}(A) = \widetilde{\mathrm{Tr}}(P_{\ker(A)}) - \widetilde{\mathrm{Tr}}(P_{\ker(A^*)}) = \widetilde{\mathrm{Tr}}\,([A, P])$, where $\mathrm{Tr} = \widetilde{\mathrm{Tr}} \circ \pi$ and $\widetilde{\mathrm{Tr}} : \Psi^{-\infty}(X, E)/[\Psi^{-\infty}(X, E), \Psi^{-\infty}(X, E)] \overset{\cong}{\to} \mathbb{C}$.

1.4. Logarithms on Categories and Cobordism

We recall that *manifold* will always stand for smooth and compact manifold (§0.1). In order to define log-functors, let us recall the definition of symmetric monoidal categories and functors.

1.4.1. Cobordism categories.

THEOREM 1.4.1 ((1.2), [**16**]). For any manifold W there exists an open *collar neighbourhood* $U \subseteq W$ of $M = \partial W$ and a diffeomorphism $h : U \to M \times [0,1)$ such that $h(m) = (m, 0)$, $\forall m \in M$.

Let $n \in \mathbb{N}$, and let W_1 and W_2 be two n-dimensional manifolds such that $\partial W_1 = M_0 \sqcup M_1$, and $\partial W_2 = M_1' \sqcup M_2$. If $f : M_1 \to M_1'$ is a diffeomorphism, then we can glue W_1 and W_2 together into a (topological) manifold $W = W_1 \cup_f W_2$. Since a smooth structure cannot be determined by the smooth structures of W_1 and W_2 alone, we need to choose collar neighbourhood for M_1 and for M_1'. In this way, W can become a (smooth) manifold with boundary $\partial W = M_0 \sqcup M_2$. Its smooth structure does depend on the choice of collar neighbourhoods, but:

THEOREM 1.4.2 (Theorem 6.3, [**57**]; Example 1.2.11, [**48**]). All smooth structures on W obtained by gluing with respect to a choice of collar neighbourhood for M_1 and M_1' are diffeomorphic. Hence, gluing of manifolds is associative up to diffeomorphism.

DEFINITION 1.4.3 (§1, [**48**]). Let $\mathbf{Cob}_n^{\mathrm{O}}$ denote the category of *unoriented cobordisms*: its objects are closed $(n-1)$-dimensional manifolds and its morphisms are *cobordisms*, i.e. equivalence classes of n-dimensional manifolds with boundary. Precisely, if $M_1, M_2 \in \mathrm{obj}(\mathbf{Cob}_n^{\mathrm{O}})$, then $\overline{W} \in \mathrm{mor}(M_1, M_2)$ is the set af all manifolds W whose boundary ∂W is diffeomorphic to $M_1 \sqcup M_2$ via a diffeomorphism $\kappa_{\partial W} : \partial W \to M_1 \sqcup M_2$ and such that $\kappa_{\partial W'}^{-1} \circ \kappa_{\partial W} : \partial W \to \partial W'$ can be extended to a diffeomorphism $W \to W'$, if W' is another such manifold. Analogously, the category of *(oriented) cobordisms* $\mathbf{Cob}_n := \mathbf{Cob}_n^{\mathrm{SO}}$ is defined as $\mathbf{Cob}_n^{\mathrm{O}}$, but this time objects and morphisms are oriented manifolds and the diffeomorphisms are orientation preserving, i.e. for $\overline{W} \in \mathrm{mor}(M_1, M_2)$ and $W \in \overline{W}$, $\kappa_{\partial W}$ is an orientation preserving diffeomorphisms from ∂W to $M_1^- \sqcup M_2$.

Let $\overline{W} \in \mathrm{mor}(M_1, M_2)$ and $\overline{W'} \in \mathrm{mor}(M_2, M_3)$, i.e. $\partial W = X^- \sqcup Y$ and $\partial W' = \widetilde{Y}^- \sqcup Z$ such that there exist diffeomorphisms $\kappa_X : X \to M_1$, $\kappa_Y : Y \to M_2$,

$\kappa_{\widetilde{Y}} : \widetilde{Y} \to M_2$, and $\kappa_Z : Z \to M_3$. If $\phi := \kappa_{\widetilde{Y}}^{-1} \circ \kappa_Y$, then composition of morphisms is defined by gluing with respect to ϕ:

$$\overline{W} \cup_\phi \overline{W'} = \overline{W \cup_\phi W'} =: \overline{W'} \circ \overline{W} \in \mathrm{mor}(M_1, M_3).$$

The identity morphism associated to $M \in \mathrm{obj}(\mathbf{Cob}_n)$ is the equivalence class of the cylinder: $\overline{M \times [0,1]} \in \mathrm{mor}(M, M)$. Clearly, $\partial(M \times [0,1]) = M^- \sqcup M$.

1.4.2. Symmetric monoidal categories and TQFT. The following definitions are taken from §2, [**72**], unless stated otherwise.

DEFINITION 1.4.4. Let $\mathbf{C}$ be a (small) category endowed with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and unit object $1_{\mathbf{C}} \in \mathrm{obj}(\mathbf{C})$ such that, for $c, c', c'' \in \mathrm{obj}(\mathbf{C})$:

$$c \otimes 1_{\mathbf{C}} \cong c \quad \text{and} \quad c \otimes (c' \otimes c'') \cong (c \otimes c') \otimes c'',$$

where $\cong$ means a coherence isomorphism. Then $\mathbf{C}$ is called *monoidal category* and $\otimes$ *monoidal product.* If also:

$$c \otimes c' \cong c' \otimes c, \tag{1.4.1}$$

then $\mathbf{C}$ is called *symmetric monoidal category* and $\otimes$ *symmetric monoidal product.*

EXAMPLE 1.4.5. $\mathbf{Cob}_n^{\mathrm{O}}$ and $\mathbf{Cob}_n$ are symmetric monoidal categories with symmetric monoidal product $\otimes := \sqcup$, the disjoint union. The unit object is the empty manifold $\emptyset$, considered as a closed $(n-1)$-dimensional manifold.

EXAMPLE 1.4.6 (Category of R-modules). For a commutative ring R, let R-**Mod** be the category with R-modules as objects and module morphisms between them. It is a symmetric monoidal category with product defined by the tensor product over R. The unit object is clearly the ring R itself. In particular, if $R = \mathbb{F}$ is a field, then $\mathbb{F}$-$\mathbf{Mod} =: \mathbf{Vect}_{\mathbb{F}}$, the category of vector spaces over $\mathbb{F}$.

REMARK 1.4.7. Since any two associativity bracketing of $x_1 \otimes \cdots \otimes x_n$, for $x_i \in \mathrm{obj}(\mathbf{C})$, coincide modulo coherence isomorphisms, we can simply write in general $x := x_1 \otimes \cdots \otimes x_n$. By (1.4.1), $\forall c, c' \in \mathrm{obj}(\mathbf{C})$ there exist braiding isomorphisms $b_{c,c'} : c \otimes c' \to c' \otimes c$, $b_{c,c'}^{-1} = b_{c,'c}$, which extend to isomorphisms $s_\sigma(x) : x \to x_\sigma := x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ for any permutation $\sigma \in S_n$.

DEFINITION 1.4.8. Let $\mathbf{C}$ be a monoidal category. Then a functor $F : \mathbf{C} \to \mathbf{B}$ is called *strict* if $F(x_1 \otimes \cdots \otimes x_n)$ is independent of the associativity bracketing and all the coherence isomorphisms are mapped into the identity in $\mathbf{B}$.

LEMMA 1.4.9 (Lemma 2.1, [**72**]). Let $\sigma \in S_n$ and $s_\sigma(x) : x \to x_\sigma$ be as in Remark 1.4.7. There exists a canonical isomorphism

$$\mu_\sigma(x) := F(s_\sigma(x)) : F(x) \to F(x_\sigma),$$

independent of associativity bracketing of x and x_σ, such that:

$$\mu_{\sigma' \circ \sigma}(x) = \mu_{\sigma'}(x_\sigma) \circ \mu_\sigma(x).$$

DEFINITION 1.4.10 (§1.1, [**48**]). Let $(\mathbf{C}, \otimes_{\mathbf{C}})$, $(\mathbf{B}, \otimes_{\mathbf{B}})$ be two symmetric monoidal categories. Then a functor $F : (\mathbf{C}, \otimes_{\mathbf{C}}) \to (\mathbf{B}, \otimes_{\mathbf{B}})$ is *symmetric monoidal* if:

$$F(\mathbf{1_C}) \cong \mathbf{1_B} \quad \text{and} \quad F(c \otimes_{\mathbf{C}} c') \cong F(c) \otimes_{\mathbf{B}} F(c'), \quad \forall c, c' \in \mathrm{obj}(\mathbf{C}).$$

Symmetric monoidal categories and functors are the necessary ingredients for the functorial definition of *Topological Quantum Field Theories* ([**2**]):

DEFINITION 1.4.11 (Definition 1.1.5, [**48**]). A *Topological Quantum Field Theory* of dimension n is a symmetric monoidal functor

$$Z : \mathbf{Cob}_n \to \mathbf{Vect}_{\mathbb{F}}.$$

Unfolding the definition, if $\overline{W} \in \mathrm{mor}(M_1, M_2)$, then $Z(\overline{W})$ is a linear map between the vector spaces $Z(M_1)$ and $Z(M_2)$, i.e.

$$Z(\overline{W}) \in \mathrm{mor}(Z(M_1), Z(M_2)) \cong Z(M_1)^* \otimes Z(M_2),$$

with $Z(M_1)^*$ the dual of $Z(M_1)$. By Proposition 1.1.8, [**48**], $Z(M_1)^* \cong Z(M_1^-)$, so $Z(\overline{W}) \in Z(M_1^-) \otimes Z(M_2) \cong Z(M_1^- \sqcup M_2)$, as Z is a symmetric monoidal functor. Hence $Z(\overline{W}) \in Z(\partial\overline{W})$ and, if $\partial\overline{W} = \emptyset$, then $Z(\overline{W}) \in \mathbb{F}$ and thus a TQFT assigns a numerical *smooth* invariant to a closed n-dimensional manifold W. In fact, W can be seen as a bordism from $\emptyset$ to itself, i.e. $\overline{W} \in \mathrm{mor}(\emptyset, \emptyset)$. Hence $Z(\overline{W}) \in \mathrm{mor}(\mathbb{F}, \mathbb{F}) \cong \mathbb{F}$. We remark that if $\overline{W}$ is a homeomorphism class, or a homotopy class, then $Z(W)$ represents *topological* or *homotopy* invariant.

1.4.3. Logarithms, traces and categories. For proofs, comments, and further examples we redirect to §2, [**72**], from which the definitions and results of this paragraph are taken, unless otherwise stated.

DEFINITION 1.4.12. The symmetric monoidal bifunctor $\otimes$ naturally defines (non-monoidal) *product functors* $\forall y \in \mathrm{obj}(\mathbf{C})$ which are respectively the right and left multiplication $m_{\otimes y}, m_{y\otimes} : \mathbf{C} \to \mathbf{C}$, i.e. $\forall c, y \in \mathrm{obj}(\mathbf{C})$, $\forall \gamma \in \mathrm{mor}(\mathbf{C})$ and $\iota_y \in \mathrm{mor}(y, y)$ the identity morphism associated to y, then

$$m_{y\otimes}c = y \otimes c \text{ and } m_{y\otimes}(\gamma) = \iota_y \otimes \gamma, \qquad m_{\otimes y}c = c \otimes y \text{ and } m_{\otimes y}(\gamma) = \gamma \otimes \iota_y.$$

Definition 1.4.13 (Monoidal product representation). Let $\mathbf{C}^*$ be a groupoid obtained from $\mathbf{C}$ by considering only a specific subclass of its isomorphisms, containing all the coherence isomorphisms and permutations s_σ. Let $\mathbf{B}$ be an additive category. Then, a functor $F : \mathbf{C}^* \to \mathbf{B}$ is called *monoidal product representation* (of the reduced category $\mathbf{C}^*$) into $\mathbf{B}$ if F is strict and $\forall\ y \in \mathrm{obj}(\mathbf{C})$ there exist a natural transformation, called *insertion transformation*

$$\eta_{\otimes y} : F \to F_{\otimes y} := F \circ m_{\otimes y}, \qquad \eta_{\otimes y}c : F(c) \to F(c \otimes y),$$

such that, $\forall c, c', y, y' \in \mathrm{obj}(\mathbf{C})$, η is:

- compatible with $\otimes$:

$$\eta_{\otimes(y\otimes y')}c = \eta_{\otimes y'}(c \otimes y) \circ \eta_{\otimes y}c \quad \text{and}$$

- compatible with the braidings $b_{c,c'}$:

$$\eta_{\otimes(y\otimes y')}c = \mu_\sigma(c \otimes y' \otimes y) \circ \eta_{\otimes(y\otimes y')}c,$$

  with σ a permutation that swaps y and y' and fixes c.

The morphisms $\eta_{\otimes y}c$ are called *insertion morphisms*.

Definition 1.4.14. A monoidal product representation is *injective* if $\eta_{\otimes y}c$ is left-invertible $\forall c, y \in \mathrm{obj}(\mathbf{C})$, i.e. there exists $\delta_{\otimes y}c \in \mathrm{mor}(F(c \otimes y), F(c))$, compatible with $\otimes$, such that $\delta_{\otimes y}c \circ \eta_{\otimes y}c = \iota_{F(c)}$. $\delta_{\otimes y}c$ is called *ejection morphisms*.

Remark 1.4.15. Insertion maps intertwine with the permutation isomorphisms (Lemma 2.4, [**72**]):

$$\eta_{\otimes y}(x_\sigma) \circ \mu_\sigma(x) = \mu_{\sigma\otimes 1}(x \otimes y) \circ \eta_{\otimes y}(x).$$

Thus, by combining insertion maps and permutation isomorphisms we obtain more general insertion maps:

$$\eta^k_y(x) : F(x_1 \otimes \cdots \otimes x_n) \to F(x_1 \otimes \cdots \otimes x_{k-1} \otimes y \otimes x_k \otimes \cdots \otimes x_n)$$
$$\eta^k_y(x) := \mu_{\sigma_{k,n+1}}(x \otimes y) \circ \eta_{\otimes y}(x),$$

with $\sigma_{k,n+1} \in S_{n+1}$ the permutation that moves y in the k^{th} position. Analogously, we can generalise the ejection morphisms in a similar fashion and obtain $\delta^k_{\otimes y}(x)$, which commute nicely with $\eta^k_y(x)$ (Lemma 2.5, [**72**]).

REMARK 1.4.16. Let $\mathrm{obj}(\mathbf{C}^p)$ denote the set of p-tuples $x_0 \otimes \cdots \otimes x_{p-1}$ of objects of $\mathbf{C}$. Then, we obtain a simplicial structure for $F(\mathbf{C}^*)$, with p-simplices $\Delta_p \subset \mathrm{obj}(\mathbf{B}) \times \mathrm{obj}(\mathbf{C}^p)$:

$$\Delta_p = \{(\xi, x_0, \dots, x_{p-1}) \mid \xi \in F(x_0 \otimes \cdots \otimes x_{p-1}), x_j \in \mathrm{obj}(\mathbf{C})\}.$$

Face maps $d_k : \Delta_p \to \Delta_{p-1}$ and degeneracy maps $s_k(w) : \Delta_p \to \Delta_{p+1}$ are respectively defined as:

$$d_k(\xi, x_0, \dots, x_{p-1}) := (\delta^k_{x_k}(\xi), x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{p-1}) \text{ and}$$
$$s_k(w)(\xi, x_0, \dots, x_{p-1}) := (\eta^k_w(\xi), x_0, \dots, x_{k-1}, w, x_k, \dots, x_{p-1}).$$

In particular, if only degeneracy maps are available, the structure is called *presimplicial.*

1.4.4. Tracial monoidal product representation. The following definitions are taken from §2.1, [**72**].

REMARK 1.4.17. If R is a ring, then the canonical projection $\pi : R \to R/[R, R]$ defines a quotient functor from the category of rings into the category of abelian groups, i.e.

$$\Pi : \mathbf{Ring} \to \mathbf{Ring}/[\mathbf{Ring}, \mathbf{Ring}] \subset \mathbf{Ab} \tag{1.4.2}$$

REMARK 1.4.18. If $(\mathbf{A}, +)$ is an additive category and $a \in \mathrm{obj}(\mathbf{A})$, then $\mathrm{end}_{\mathbf{A}}(a) := \mathrm{mor}_{\mathbf{A}}(a, a)$ is a ring, the product being the composition. In particualr, if $\mathbf{A} = R\text{-}\mathbf{Mod}$, then R-linearity of the morphisms yields that $\mathrm{end}_{\mathbf{A}}(a)$ is an R-algebra.

DEFINITION 1.4.19. A monoidal product representation $F : \mathbf{C} \to \mathbf{Ring}$ of a symmetric monoidal category $\mathbf{C}$ is said to be *pretracial* with respect to an additive category $\mathbf{A}$ if:

- $\forall c \in \mathrm{obj}(\mathbf{C}) \ \exists\, ! \ a_c \in \mathrm{obj}(\mathbf{A})$ such that $F(c) = \mathrm{end}_{\mathbf{A}}(a_c)$;
- $\eta_{\otimes y} c$ are ring homomorphisms;
- $\mu_\sigma c$ are ring isomorphisms.

Then we will write $F : \mathbf{C}^* \to \mathbf{Ring}_{\mathbf{Add}}$. Moreover, if $\delta_{\otimes y} c$ preserves commutators, i.e. $\delta_{\otimes y} c([F(c \otimes y), F(c \otimes y)]) \subset [F(c), F(c)]$, then F is called *injective.*

LEMMA 1.4.20. Let $F : \mathbf{C}^* \to \mathbf{Ring}_{\mathbf{Add}}$ be a pretracial monoidal product representation. Then by compositon with (1.4.2), the functor

$$F_\Pi =: \Pi \circ F : \mathbf{C}^* \to F_\Pi(\mathbf{C}^*) := F(\mathbf{C}^*)/[F(\mathbf{C}^*), F(\mathbf{C}^*)] \subset \mathbf{Ab}$$

is a monoidal product representation with insertion homomorphisms

$$\widetilde{\eta}_{\otimes y}c : \frac{F(c)}{[F(c), F(c)]} \to \frac{F(c \otimes y)}{[F(c \otimes y), F(c \otimes y)]}$$

and $(F_\Pi(\mathbf{C}^*), \widetilde{\eta}_y^k)$ inherits the structure of a presimplicial set.

Definition 1.4.21. A symmetric monoidal category $\mathbf{C}$ has a *categorical trace* τ if there exist elements $c \in \mathrm{obj}(\mathbf{C})$ for which we have a non-empty subclass $\mathrm{end}^\tau_{\mathbf{C}}(c) \subset \mathrm{end}_{\mathbf{C}}(c)$ and a map $\tau_c : \mathrm{end}^\tau_{\mathbf{C}}(c) \to \mathrm{end}_{\mathbf{C}}(1_{\mathbf{C}})$ such that the following *trace property* holds: $\forall \alpha \in \mathrm{mor}(c, c')$, $\beta \in \mathrm{mor}(c', c)$ such that $\beta \circ \alpha \in \mathrm{end}^\tau_{\mathbf{C}}(c)$ and $\alpha \circ \beta \in \mathrm{end}^\tau_{\mathbf{C}}(c')$,

$$\tau_c(\beta \circ \alpha) = \tau_{c'}(\alpha \circ \beta).$$

Elements $\alpha \in \mathrm{end}^\tau_{\mathbf{C}} c$ are said to be τ-trace class.

Example 1.4.22. R-**Mod** of Example 1.4.6 is trace class, since $M_{m\times n}(R)$, the algebra of matrices with R coefficients, has a (classical) trace. More interestingly, $\mathbf{Cob}_n$ is trace class, with trace sending $\overline{W} \in \mathrm{mor}(M, M)$ to the closed n-manifold obtained by gluing the boundary together.

Definition 1.4.23. A pretracial monoidal product representation of $\mathbf{C}$, $F : \mathbf{C}^* \to \mathbf{Ring_{Add}}$, is called *tracial* if the background additive category $\mathbf{A}$ has an F-compatible trace τ, i.e. the ring homomorphisms $\tau_c : F(c) = \mathrm{end}_{\mathbf{A}}(a_c) \to \mathrm{end}_{\mathbf{A}}(1)$ satisfy $\tau_{c\otimes y} \circ \eta_{\otimes y} c = \tau_c$ and $\tau_{x_\sigma} \circ \mu_\sigma(x) = \tau_x$.

Remark 1.4.24. In a tracial monoidal product representation, τ_c factors though $\pi_c : F(c) \to F(c)/[F(c), F(c)]$, i.e. $\tau_c = \widetilde{\tau}_c \circ \pi_c$. Moreover, the trace $\widetilde{\tau}$ on $F_\Pi(\mathbf{C}^*)$ satisfies an analogous compatibility condition:

$$\widetilde{\tau}_{c\otimes y} \circ \widetilde{\eta}_{\otimes y} c = \widetilde{\tau}_c.$$

1.4.5. Logarithmic functors. The following definitions are taken from §2.2, [**72**]. Specific references are provided when needed.

Definition 1.4.25. The *nerve* $\mathcal{N}\mathbf{C}$ of a category $\mathbf{C}$ is a simplicial set with p-simplices defined as p-tuples of morphisms:

$$(\alpha_0, \ldots, \alpha_{p-1}), \qquad \alpha_j \in \mathrm{mor}(x_j, x_{j+1}),\ j \in \{0, \ldots, p-1\}.$$

The set of all p-simplices is denoted by $\mathcal{N}_p\mathbf{C}$, and face maps $d_j : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p-1}\mathbf{C}$ and degeneracy maps $s_j : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p+1}\mathbf{C}$ are respectively defined as:

$$d_j(\alpha_0, \ldots, \alpha_{j-1}, \alpha_j, \ldots, \alpha_{p-1}) := (\alpha_0, \ldots, \alpha_j \circ \alpha_{j-1}, \ldots, \alpha_{p-1}) \text{ and}$$

$$s_j(\alpha_0, \ldots, \alpha_{j-1}, \alpha_j, \ldots, \alpha_{p-1}) := (\alpha_0, \ldots, \alpha_{j-1}, \iota_{x_j}, \alpha_j, \ldots, \alpha_{p-1}).$$

EXAMPLE 1.4.26. $\mathcal{N}_0\mathbf{C} = \mathrm{obj}(\mathbf{C})$ and $\mathcal{N}_1\mathbf{C} = \mathrm{mor}(\mathbf{C})$.

REMARK 1.4.27. We recall that, if (X, d_j, s_j) and (Y, d'_j, s'_j) are simplicial sets, a *simplicial map* $f : X \to Y$ consists of a family of maps that commute with the face and degeneracy maps, i.e. $f_p : \Delta_p \to \Delta'_p$ such that

$$f_{p-1}d_j = d'_j f_p \qquad \text{and} \qquad f_p s_j = s'_j f_{p-1}. \tag{1.4.3}$$

If Y is only *presimplicial*, i.e. there are no face maps d'_j, then $f : (X, d_j, s_j) \to (Y, s'_j)$ is said to be a *presimplicial map* if $s'_j f_{p-1} d_j = f_p$, which implies (1.4.3) when f is simplicial.

As $\left(F_\Pi(\mathbf{C}^*), \widetilde{\eta}^k_y\right)$ is a presimplicial set by Lemma 1.4.20, we can finally define:

DEFINITION 1.4.28 (Definition 2.13, [**72**]). Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category and $F : \mathbf{C}^* \to \mathbf{Ring_{Add}}$ a pretracial monoidal product representation. Then a *logarithmic functor*, or *log-functor*, is a presimplicial log-additive map

$$\log : (\mathcal{N}\mathbf{C}, d_j, s_j) \to \left(F_\Pi(\mathbf{C}^*), \widetilde{\eta}^j\right),$$

which is said to define a logarithmic representation of $\mathbf{C}$. In other words, a log-functor is a simplicial system on $\mathcal{N}_1\mathbf{C}$ of maps

$$\log_{x\otimes y} : \mathrm{mor}(x, y) \to \frac{F(x \otimes y)}{[F(x \otimes y), F(x \otimes y)]}, \quad \alpha \mapsto \log_{x\otimes y} \alpha, \quad x, y \in \mathrm{obj}(\mathbf{C}) \setminus 1_{\mathbf{C}}$$

such that if $\alpha \in \mathrm{mor}(x, y)$ and $\beta \in \mathrm{mor}(y, z)$, then

$$\log_{x\otimes y\otimes z}(\alpha, \beta) = \widetilde{\eta}_{\otimes z}(\log_{x\otimes y} \alpha) + \widetilde{\eta}_{x\otimes}(\log_{y\otimes z} \beta) \tag{1.4.4}$$

in $F(x \otimes y \otimes z)/[F(x \otimes y \otimes z), F(x \otimes y \otimes z)]$, that is

$$\log_{x\otimes y\otimes z}(\alpha, \beta) = \eta_{\otimes z}(\log_{x\otimes y} \alpha) + \eta_{x\otimes}(\log_{y\otimes z} \beta) + \sum_{j=1}^{m} [v_j, v_j] \in F(x \otimes y \otimes z).$$

On the other hand, since

$$\log_{x\otimes y\otimes z}(\alpha, \beta) = \widetilde{\eta}_y(\log_{x\otimes z} \beta \circ \alpha) \in \frac{F(x \otimes y \otimes z)}{[F(x \otimes y \otimes z), F(x \otimes y \otimes z)]},$$

(1.4.4) is equivalent to:

$$\widetilde{\eta}_y(\log_{x\otimes z} \beta \circ \alpha) = \widetilde{\eta}_{\otimes z}(\log_{x\otimes y} \alpha) + \widetilde{\eta}_{x\otimes}(\log_{y\otimes z} \beta). \tag{1.4.5}$$

REMARK 1.4.29. In Definition 1.4.28 it is enough to specify the maps on $\mathcal{N}_1\mathbf{C}$ and it suffices to define (1.4.5) (Lemma 2.16, [**72**]), since all the other simplicial maps, i.e. those on $\mathcal{N}_p\mathbf{C}$, $p \geq 1$, depend on those on $\mathcal{N}_1\mathbf{C}$. Moreover, from the definition one has all the other properties of logarithms, e.g. the log of an idempotent object is trivial. For a complete description, see Lemma 2.18, [**72**].

DEFINITION 1.4.30. Let F be a tracial monoidal product representation of a symmetric monoidal category $\mathbf{C}$, with τ the trace. Then the τ-character of the log-functor defines a *log-determinant functor representation* of $\mathbf{C}$, i.e. $\forall\, \alpha \in \mathrm{mor}_{\mathbf{C}}(c, c')$:

$$\widetilde{\tau}(\log \alpha) := \widetilde{\tau}_{c\otimes c'} \circ \log_{c\otimes c'} \alpha \in \mathrm{end}_{\mathbf{A}}(1).$$

REMARK 1.4.31. By Remark 1.4.24, we have that the log-determinant representation is independent of insertion maps (of any order: see Lemma 2.19, [**72**]):

$$\widetilde{\tau}_{c\otimes c'}(\log_{c\otimes c'} \alpha) = \widetilde{\tau}_{c\otimes c'\otimes y}(\log_{c\otimes c'\otimes y} \alpha) \tag{1.4.6}$$

Hence a log-determinant is independent of where it is computed (Lemma 2.20, [**72**]):

$$\widetilde{\tau}(\log \beta\alpha) = \widetilde{\tau}(\log \alpha) + \widetilde{\tau}(\log \beta), \qquad \alpha \in \mathrm{mor}(c, c'), \beta \in \mathrm{mor}(c', c'').$$

REMARK 1.4.32. A log-functor can be extended to elements $\delta \in \mathrm{mor}_{\mathbf{C}}(1, 1)$. In fact, after choosing $\alpha \in \mathrm{mor}_{\mathbf{C}}(1, z)$ and $\beta \in \mathrm{mor}_{\mathbf{C}}(z, 1)$ such that $z \neq 1$ and $\delta = \beta \circ \alpha$, we can define:

$$\log_z \delta := \log_{1\otimes z\otimes 1}(\alpha, \beta) \in \frac{F(1 \otimes z \otimes 1)}{[F(1 \otimes z \otimes 1), F(1 \otimes z \otimes 1)]} \cong \frac{F(z)}{[F(z), F(z)]}.$$

It depends on δ and z, but by Lemma 2.19, [**72**], not on α or β. Moreover, if a categorical trace τ is defined, then the corresponding log-determinant

$$\widetilde{\tau}(\log_z \delta) = \widetilde{\tau}(\log_z \alpha) + \widetilde{\tau}(\log_z \beta)$$

depends only on δ, by Lemma 2.20, [**72**].

1.4.6. Logarithmic Topological Quantum Field Theories. The following definitions are taken from §2–§3, [**72**]. Specific references are provided when needed.

DEFINITION 1.4.33. Let $M := M_1 \sqcup \cdots \sqcup M_p \in \mathrm{obj}(\mathbf{Cob}_n^p)$, where M_j may also be disconnected. If we write M^- for M with some of its connected components chosen with opposite orientation, then a pretracial monoidal product representation $F : \mathbf{Cob}_n^* \to \mathbf{Ring}_{\mathbf{Add}}$ is called *unoriented* if $F(M^-) = F(M)$.

DEFINITION 1.4.34. Let $F : \mathbf{Cob}_n^* \to \mathbf{Ring}_{\mathbf{Add}}$ be an unoriented pretracial monoidal product representation. Then a *Logarithmic Topological Quantum Field Theory* relative to F of dimension n, or *LogTQFT*, is a log-functor

$$\log : (\mathcal{N}\mathbf{Cob}_n, d_j, s_j) \to \left(F_\Pi(\mathbf{Cob}_n^*), \widetilde{\eta}^k_{\sqcup \cdot}\right).$$

By definition, this is a simplicial system of logarithms

$$\log_{M_1 \sqcup M_2} : \mathrm{mor}(M_1, M_2) \to F_\Pi(M_1 \sqcup M_2)$$

and a logarithm $\log_{M_1 \sqcup M_2} \overline{W} \in F(M_1 \sqcup M_2) = F(M_1^- \sqcup M_2)$ is identified to an element $\log_{\partial W} \overline{W} \in F_\Pi(\partial W)$, since $F(\partial W) \cong F(M_1 \sqcup M_2)$.

REMARK 1.4.35. Even in the case that F is unoriented, $\log_{\partial W} \overline{W}$ could depend on the orientation of $\overline{W}$. Therefore, in the case that $\log_{\partial W} \overline{W} = \log_{\partial W^-} \overline{W}^-$ for all $\overline{W}$, the LogTQFT is called *unoriented*. An example is provided by the (relative) Euler characteristic (see Chapter 2), while the topological signature is an example of a log-character of a LogTQFT that is not unoriented.

PROPOSITION 1.4.36 (Proposition 2.18, [**72**]). Let $C_M = \overline{M \times [0,1]}$ be the cobordism class of the cylinder. Then

$$\widetilde{\eta}_M \log_{M \sqcup M} C_M = 0 \;\in\; F_\Pi(M \sqcup M \sqcup M)$$

and if F is injective, then $\log_{M \sqcup M} C_M = 0$ in $F_\Pi(M \sqcup M)$.

A LogTQFT can define a TQFT, at least in a week sense:

LEMMA 1.4.37 (Lemma 3.4, [**72**]). Let $F : \mathbf{Cob}_n^* \to \mathbf{Ring}_{\mathbf{Add}}$ be an unoriented tracial monoidal product representation with trace $\tau_c : \mathrm{end}_{\mathbf{A}}(a_c) \to \mathrm{end}_{\mathbf{A}}(1)$ and let $\log : \mathcal{N}\mathbf{Cob}_n \to F_\Pi(\mathbf{Cob}_n^*)$ be a LogTQFT relative to F of dimension n. If $\epsilon : \mathrm{end}_{\mathbf{A}}(1) \to \mathbb{F}$ is an exponential map into a field, then there exists a scalar-valued symmetric monoidal functor $Z_{\log,\tau,\epsilon}$, i.e. a TQFT, defined as follows:

$$Z_{\log,\tau,\epsilon}(M) = \mathbb{F} \qquad \text{and} \qquad Z_{\log,\tau,\epsilon}(\overline{W}) = \epsilon(\tau(\log \overline{W})).$$

The following fundamental example of unoriented tracial monoidal product representation can be found in §2.1.2, [**72**], and will be useful in the next chapters.

EXAMPLE 1.4.38. Let $\mathbb{C}$-**Alg** is the category of $\mathbb{C}$-algebras and consider the strict functor $F_{-\infty} : \mathbf{Cob}_n^* \to \mathbb{C}\text{-}\mathbf{Alg}$ defined as:

$$M \in \mathrm{obj}(\mathbf{Cob}_n) \mapsto F_{-\infty}(M) := \Psi^{-\infty}(M, E),$$

for $E \to M$ some vector bundle. It comes with insertion maps:

$$\eta_N := \eta_{\sqcup N} : F_{-\infty}(M) \hookrightarrow F_{-\infty}(M_N) \qquad \eta_N(T) = j_N^* \circ T \circ i_N^*,$$

where $M := M_1 \sqcup \cdots \sqcup M_l$, $M_N := M_1 \sqcup \cdots \sqcup N \sqcup \cdots \sqcup M_l$, and $j_N^* : \Omega(M) \to \Omega(M_N)$ and $i_N^* : \Omega(M_N) \to \Omega(M)$ are the pull-backs of the projection $j_N : M_N \to M$ and the inclusion $i_N : M \hookrightarrow M_N$, respectively. Hence $F_{-\infty}$ is pretracial, but not injective, and pushes down to:

$$F_{-\infty,\Pi} : \mathbf{Cob}_n^* \to F_{-\infty,\Pi}(\mathbf{Cob}_n^*)$$

with insertion maps

$$\widetilde{\eta}_N(M) : \frac{F_{-\infty}(M)}{[F_{-\infty}(M), F_{-\infty}(M)]} \to \frac{F_{-\infty}(M_N)}{[F_{-\infty}(M_N), F_{-\infty}(M_N)]}.$$

Let $\mathrm{Tr}_M : F_{-\infty}(M) \to \mathbb{C}$ the classical trace on smoothing ψdos (0.2.1). Since Tr_M is the unique trace on $F_{-\infty}(M)$ (Lemma 2.10, [**72**]), by Lemma 1.2.4 there exists $\widetilde{\mathrm{Tr}}_M : \pi_M(F_{-\infty}(M)) \overset{\cong}{\to} \mathbb{C}$ such that:

$$\mathrm{Tr}_M = \widetilde{\mathrm{Tr}}_M \circ \pi_M, \quad \mathrm{Tr}_M = \mathrm{Tr}_{M_N} \circ \eta_N \quad \text{and} \quad \widetilde{\mathrm{Tr}}_M = \widetilde{\mathrm{Tr}}_M \circ \widetilde{\eta}_N. \tag{1.4.7}$$

Hence $(F_{-\infty}, \mathrm{Tr})$ is a tracial monoidal product representation.

Lemma 1.4.39 (Lemma 2.12, [**72**]).

- $(F_{-\infty}, \mathrm{Tr})$ is an *unoriented* tracial monoidal product representation;
- a diffeomorphism $\phi : M \to N$, $M, N \in \mathrm{obj}(\mathbf{Cob}_n)$, induces a canonical continuous isomorphism of algebras:

$$\phi_\sharp : F_{-\infty}(M) \to F_{-\infty}(N) \ \text{ such that } \ \mathrm{Tr}_M = \mathrm{Tr}_N \circ \phi_\sharp; \tag{1.4.8}$$

- ϕ pushes-down to a canonical linear isomorphism of complex lines:

$$\vartheta_{M,N} : \frac{F_{-\infty}(M)}{[F_{-\infty}(M), F_{-\infty}(M)]} \to \frac{F_{-\infty}(N)}{[F_{-\infty}(N), F_{-\infty}(N)]} \tag{1.4.9}$$

  which is independent of the initial ϕ.

1.4.7. The Unoriented Logarithm Theorem for Surfaces. We conclude this chapter with a novel result for LogTQFTs of dimension 2, i.e. on compact oriented surfaces. We shall see that an unoriented LogTQFT is characterised by its definition on the unit disc D. First, we prove it for closed compact surfaces. The general case will follow as a Corollary.

THEOREM 1.4.40. Let $F : \mathbf{Cob}_2^* \to \mathbf{Ring}$ be an injective and unoriented monoidal product representation and let $\log : \mathcal{N}\mathbf{Cob}_2 \to (F_\Pi(\mathbf{Cob}_2^*), \widetilde{\eta})$ be an unoriented LogTQFT. Let Σ_g denote an orientable, closed and connected surface of genus g and $\chi(\Sigma_g) = 2 - 2g$ its Euler characteristic. Then, if D denotes the unit disc,

$$\begin{aligned} &\log_{S^1} \overline{\Sigma}_0 = \chi(\Sigma_0) \cdot \log_{S^1} \overline{D} && \text{for } g = 0, \\ &\log_{S^1 \sqcup S^1} \overline{\Sigma}_1 = \chi(\Sigma_1) \cdot \widetilde{\eta}_{S^1} \log_{S^1} \overline{D} && \text{for } g = 1 \text{ and} \\ &\log_{S^1 \sqcup S^1 \sqcup S^1} \overline{\Sigma}_g = \chi(\Sigma_g) \cdot \widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D}, && \text{for any } g > 1. \end{aligned}$$

PROOF. We start with some observations. If we consider the unit disc D to be a morphism $\emptyset \to S^1$, then $D^- : S^1 \to \emptyset$. Thus since F and log are unoriented, $\log_{S^1} \overline{D} = \log_{S^1} \overline{D^-} \in F_\pi(S^1) := F(S)/[F(S), F(S)]$. In the same way, we can see the *pair of pants* to be a morphism $P : S^1 \to S^1 \sqcup S^1$. Hence $P^- : S^1 \sqcup S^1 \to S^1$ and $\log_{S^1 \sqcup S^1 \sqcup S^1} \overline{P} = \log_{S^1 \sqcup S^1 \sqcup S^1} \overline{P^-} \in F_\pi(S^1 \sqcup S^1 \sqcup S^1)$.

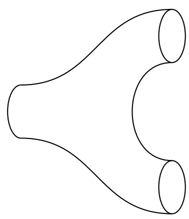

FIGURE 1. The *pair of pants.*

Finally, let us consider the cylinder $C = S^1 \times [0,1] : S^1 \to S^1$. On the one hand, C corresponds to a map $\widetilde{C} : \emptyset \to {S^1}^- \sqcup S^1$, but both surfaces are diffeomorphic, so they are accounted for in the same cobordism $\overline{C}$. On the other hand, $C^- : {S^1}^- \to {S^1}^-$ is diffeomorphic to $\widetilde{C}^- : S^1 \sqcup {S^1}^- \to \emptyset$, thus they define the same cobordims and since log is unoriented and F injective, we conclude $\log_{S^1 \sqcup S^1} \overline{C^-} = \log_{S^1 \sqcup S^1} \overline{C} = 0 \in F_\pi(S^1 \sqcup S^1)$.

Now, since $\overline{\Sigma}_g \in \mathrm{mor}_{\mathbf{Cob}_2}(\emptyset, \emptyset)$, its logarithm must be defined relative to a choice of embedded closed curve $S \in \mathrm{obj}(\mathbf{Cob}_2)$:

$$\log_S \overline{\Sigma}_g := \log_S(\emptyset \xrightarrow{\alpha} S \xrightarrow{\beta} \emptyset) \in F_\pi(S)$$

and depends on S, but not on the particular α, β used (Remark 1.4.32). So if $g = 0$, $\Sigma_0 = S^2$ is the 2-sphere and the easiest choice for S is the unit circle S^1. Hence $\log_{S^1} \overline{S^2} = \log_{S^1}(\emptyset \xrightarrow{\overline{D}} S^1 \xrightarrow{\overline{D^-}} \emptyset)$ and by (1.4.5)

$$\log_{S^1}(\emptyset \xrightarrow{\overline{D}} S^1 \xrightarrow{\overline{D^-}} \emptyset) = \log_{S^1} \overline{D} + \log_{S^1} \overline{D^-} = 2 \cdot \log_{S^1} \overline{D} = \chi(\Sigma_0) \cdot \log_{S^1} \overline{D}.$$

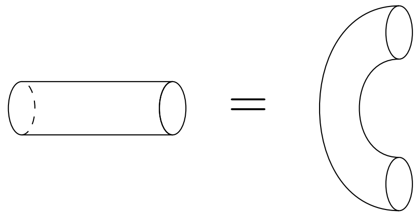

FIGURE 2. Dual interpretation of $\overline{C}$.

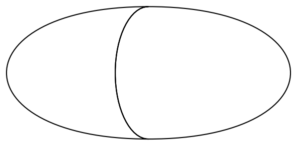

FIGURE 3. $\overline{S^2}$ as $\overline{D} \cup_{S^1} \overline{D^-}$.

Analogously, let $g = 1$, so $\Sigma_1 = T^2$ is the 2-torus. Then we can split it into two cylinders, $\emptyset \xrightarrow{\overline{C}} {S^1}^- \sqcup S^1$ and $S^1 \sqcup {S^1}^- \xrightarrow{\overline{C^-}} \emptyset$, and obtain:

$$\log_{S^1 \sqcup S^1} \overline{T^2} = \log_{S^1 \sqcup S^1} \overline{C} + \log_{S^1 \sqcup S^1} \overline{C^-} = 0 = \chi(T^2) \cdot \widetilde{\eta}_{S^1} \log_{S^1} \overline{D}.$$

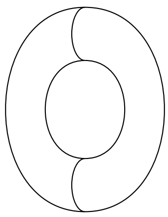

FIGURE 4. $\overline{T^2}$ as $\overline{C} \cup_{S^1 \sqcup S^1} \overline{C^-}$.

In general, let Σ_g be any closed and connected surface with $g > 1$. Then we can split $\emptyset \xrightarrow{\overline{\Sigma_g}} \emptyset$ into $2g$ pair of pants and 2 discs:

$$\emptyset \xrightarrow{\overline{D}} \underbrace{S^1 \xrightarrow{\overline{P}} S^1 \sqcup S^1 \xrightarrow{\overline{P^-}} S^1 \xrightarrow{\overline{P}} S^1 \sqcup S^1 \xrightarrow{\overline{P^-}} S^1 \xrightarrow{\overline{P}} \cdots \xrightarrow{\overline{P^-}} S^1 \xrightarrow{\overline{P}} S^1 \sqcup S^1 \xrightarrow{\overline{P^-}} S^1}_{2g} \xrightarrow{\overline{D^-}} \emptyset.$$

Since $\partial \overline{P} = {S^1}^- \sqcup S^1 \sqcup S^1$, it suffices to embed all logarithms into $F_\pi(S^1 \sqcup S^1 \sqcup S^1)$. Hence:

$$\begin{aligned} \log_{S^1 \sqcup S^1 \sqcup S^1} \overline{\Sigma}_g &= \log_{S^1 \sqcup S^1 \sqcup S^1} (\emptyset \xrightarrow{\overline{D}} S^1 \xrightarrow{\overline{P}} S^1 \sqcup S^1 \xrightarrow{\overline{P^-}} \cdots \xrightarrow{\overline{P}} S^1 \sqcup S^1 \xrightarrow{\overline{P^-}} S^1 \xrightarrow{\overline{D}} \emptyset) \\ &= 2\widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D} + 2g \cdot \log_{S^1 \sqcup S^1 \sqcup S^1} \overline{P}. \end{aligned} \tag{1.4.10}$$

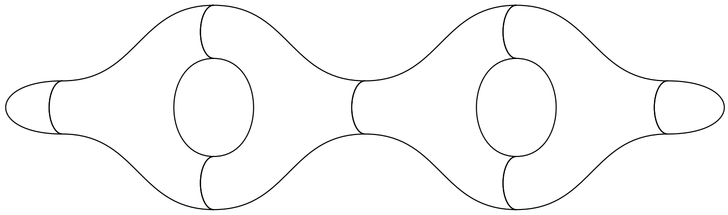

FIGURE 5. $\overline{\Sigma}_g$ for $g = 2$.

Since the cylinder can be split into a disc and a pair of pants:

$$\emptyset \xrightarrow{\overline{C}} S^1 \sqcup S^1 = \emptyset \xrightarrow{\overline{D}} S^1 \xrightarrow{\overline{P}} S^1 \sqcup S^1,$$

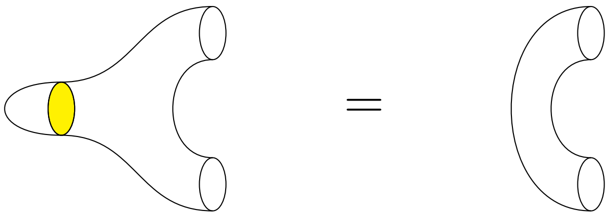

FIGURE 6. $\overline{D} \cup_{S^1} \overline{P} = \overline{C}$.

we have that $0 = \widetilde{\eta}_{S^1} \log_{S^1 \sqcup S^1} \overline{C} = \widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D} + \log_{S^1 \sqcup S^1 \sqcup S^1} \overline{P}$, which yields $\log_{S^1 \sqcup S^1 \sqcup S^1} \overline{P} = -\widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D}$ and (1.4.10) becomes

$$\log_{S^1 \sqcup S^1 \sqcup S^1} \overline{\Sigma}_g = (2 - 2g)\, \widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D} = \chi(\Sigma_g) \cdot \widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D}.$$

□

REMARK 1.4.41. The injectivity hypothesis for F (Proposition 1.4.36) can be safely relaxed. In that case we obtain $\widetilde{\eta}_{S^1} \log_{S^1 \sqcup S^1} \overline{\Sigma}_1 = \chi(\Sigma_1) \cdot \widetilde{\eta}_{S^1 \sqcup S^1} \log_{S^1} \overline{D} (= 0)$.

COROLLARY 1.4.42 (Unoriented Logarithm Theorem for Orientable Surfaces). Let $F : \mathbf{Cob}_2^* \to \mathbf{Ring}$ be an injective and unoriented monoidal product representation and let $\log : \mathcal{N}\mathbf{Cob}_2 \to (F_\Pi(\mathbf{Cob}_2^*), \widetilde{\eta})$ be an unoriented LogTQFT. Let $\Sigma_{g,k}$ denote an orientable, compact, and connected surface of genus g, whose boundary $\partial \Sigma_{g,k}$ has k connected components, i.e. $\partial \Sigma_{g,k} \cong \bigsqcup_k S^1$. Then, $\forall g, k \in \mathbb{N}$:

$$\log_{\bigsqcup_k S^1} \overline{\Sigma}_{g,k} = \chi(\Sigma_{g,k}) \cdot \widetilde{\eta}_{\bigsqcup_{j=1}^{k-1} S^1} \log_{S^1} \overline{D}, \tag{1.4.11}$$

where $\chi(\Sigma_{g,k}) = \chi(\Sigma_g) - k$ is the Euler characteristic of $\Sigma_{g,k}$ and $\chi(\Sigma_g)$ is the closed surface Σ_g obtained from $\Sigma_{g,k}$ by gluing k discs along the boundary components.

PROOF. We prove the statement by induction on k. If $k=0$, then the statement corresponds to Theorem 1.4.40, so let us assume the statement true for $k \leq n$. Since the surface $\Sigma_{g,n}$ has boundary $\partial\Sigma_{g,n} \cong \bigsqcup_n S^1$ and defines a cobordism $\bigsqcup_n S^1 \overset{\overline{\Sigma}_{g,n}}{\to} \emptyset$, it can be decomposed as $\bigsqcup_n S^1 \overset{\overline{\Sigma}_{g,n+1}}{\to} S^1 \overset{\overline{D}}{\to} \emptyset$. Thus, by (1.4.5):

$$\widetilde{\eta}_{S^1} \log_{\bigsqcup_n S^1} \overline{\Sigma}_{g,n} = \log_{\bigsqcup_{n+1} S^1} \overline{\Sigma}_{g,n+1} + \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1} \log_{S^1} \overline{D}.$$

Thence, by inductive hypothesis:

$$\begin{aligned}
\log_{\bigsqcup_{n+1} S^1} \overline{\Sigma}_{g,n+1} &= \widetilde{\eta}_{S^1} \log_{\bigsqcup_n S^1} \overline{\Sigma}_{g,n} - \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1} \log_{S^1} \overline{D} \\
&= \chi(\Sigma_{g,n}) \cdot \widetilde{\eta}_{S^1} \widetilde{\eta}_{\bigsqcup_{j=1}^{n-1} S^1} \log_{S^1} \overline{D} - \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1} \log_{S^1} \overline{D} \\
&= (\chi(\Sigma_g) - n) \cdot \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1} \log_{S^1} \overline{D} - \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1} \log_{S^1} \overline{D} \\
&= (\chi(\Sigma_g) - (n+1)) \cdot \widetilde{\eta}_{\bigsqcup_{j=1}^k S^1} \log_{S^1} \overline{D},
\end{aligned}$$

where $\widetilde{\eta}_{S^1} \widetilde{\eta}_{\bigsqcup_{j=1}^{n-1} S^1} = \widetilde{\eta}_{\bigsqcup_{j=1}^n S^1}$.

□

REMARK 1.4.43. Let $F : \mathbf{Cob}_1^* \to \mathbf{Ring}$ be an injective and unoriented monoidal product representation and let $\log : \mathcal{N}\mathbf{Cob}_1 \to (F_\Pi(\mathbf{Cob}_1^*), \widetilde{\eta})$ be an unoriented LogTQFT of dimension 1. Then $M \in \mathrm{obj}\mathbf{Cob}_1)$ is a collection of points and $\overline{W} \in \mathrm{mor}(M_1, M_2)$ is a disjoint union of line segments $L = \{\mathrm{pt}\} \times [0,1]$. Hence, by the same approach of Theorem 1.4.40, every unoriented LogTQFT is trivial and in particular $\log \overline{S^1} = 0$. This, together with an exponential map $\epsilon : \mathrm{end}_{\mathbf{A}}(1) \to \mathbb{F}$, can give rise to a (rather trivial) 1-dimensional TQFT as described in Lemma 1.4.37, where $Z_{\log,\tau,\epsilon}(\overline{W}) = 1 \in \mathbb{F}$. In particular, this gives $Z_{\log,\tau,\epsilon}(S^1) = 1$, which thus retrieves the dimension of the vector space assigned to a point (which is $\mathbb{F}$ itself), as prescribed by Lurie in Example 1.1.9 of [**48**]. Less trivial TQFTs can be obtained dropping the unoriented hypothesis.

CHAPTER 2

Dirac operators and Logarithms

In this chapter we will show how the relative (or absolute) Euler characteristic of an even dimensional manifold with boundary can be realised as a log-determinant of a LogTQFT. The idea is similar to the proof of the same fact for the topological signature (done in [**72**]) and relies on index theory of Elliptic Boundary Value Problems.

Since Index Theory will have a key role also in Part II (with appropriate generalizations), we will recall the main definitions, such as the realization of Dirac operator with respect to well-posed boundary conditions, the APS Index Theorem and the quasi-additive formula of the index. In particular, we will prove this formula again, but from the point of view of Calderón projectors. To our knowledge, this has not been done.

2.1. Dirac operators

Let $E \to X$ be a complex vector bundle, with X an n-dimensional manifold with (possibly empty) boundary $Y := \partial X$. The following definitions are taken from §3, §8, and §14, [**10**]. Specific references are provided when needed.

DEFINITION 2.1.1. A *Dirac-type operator* is a first order differential operator $D : C^\infty(X,E) \to C^\infty(X,E)$ such that the principal symbol of D^2 defines the Riemannian metric, i.e $\sigma^{D^2}(x,\xi) = \sum_{i,j=1}^n g_X^{ij}(x)\xi_i\xi_j$. D^2 is called *Dirac Laplacian.*

In addition, let $E \to X$ be a Clifford bundle, $\mathbf{c} : C^\infty(X, TX \otimes E) \to C^\infty(X,E)$ be the left Clifford multiplication and $J : C^\infty(X, T^*X \otimes E) \to C^\infty(X, TX \otimes E)$ the isomorphisms between vector and covector fields. Then the first order differential operator

$$\eth := \mathbf{c} \circ J \circ \nabla^E : C^\infty(X,E) \to C^\infty(X,E)$$

is called *generalised Dirac operator.* $\eth$ and $\eth^2$ are elliptic with principal symbols (Lemma 3.3, [**10**]):

$$\sigma^{\eth}(x,\xi) = i\mathbf{c}(\xi) : E_x \to E_x \quad \text{and} \quad \sigma^{\eth^2}(x,\xi) = \|\xi\|^2 I : E_x \to E_x.$$

Moreover, if ∇^E is *compatible*[1] with the Clifford module structure of E we call $\eth$ a *(compatible) Dirac operator*.

EXAMPLE 2.1.2. The de Rham operator $d+\delta : \Omega(X) \to \Omega(X)$ in §0.3 is a compatible Dirac operator with $(d+\delta)^2 = \Delta$, i.e. the Hodge-Laplacian (Definition 0.3.1). Since $\mathbf{c}(\xi) = \mathrm{ext}(\xi) - \mathrm{int}(\xi)$ when $E = \Lambda(X)$, i.e. the difference between *exterior* and *interior multiplication*, we have $\sigma^{d+\delta}(x,\xi) = i\mathbf{c}(\xi) = i(\mathrm{ext}(\xi) - \mathrm{int}(\xi))$ (Lemma 1.5.3, [**23**]).

THEOREM 2.1.3. Dirac operators satisfy:

i) the *Unique Continuation Property*:

'If a solution s of $\eth s = 0$ vanishes in an open subset of X, then $s = 0$ on the whole connected component of X';

ii) *Green's formula*:

$$\langle \eth s_1, s_2 \rangle_X - \langle s_1, \eth s_2 \rangle_X = -\langle \sigma\gamma s_1, \gamma s_2 \rangle_Y, \quad s_1, s_2 \in C^\infty(X,E).$$

with $\sigma = \mathbf{c}(dt) : E_{|U} \to E_{|U}$ the Clifford multiplication by the inward unit normal. In particular, $\eth$ is formally self-adjoint in the interior of X, i.e. $\langle \eth s_1, s_2 \rangle_X = \langle s_1, \eth s_2 \rangle_X$ if $s_1, s_2 \in C^\infty(X,E)$ with support disjoint from Y.

REMARK 2.1.4. $\sigma : E_{|U} \to E_{|U}$, called *Green's form* of $\eth$, is constant in t and is skew-adjoint, i.e. $\sigma^* = \sigma^{-1} = -\sigma$.

REMARK 2.1.5 (§3, [**25**]). Any first order elliptic differential operator can be represented on a collar neighbourhood U of Y as $\Sigma(\partial_t + \mathcal{B}_t)$, where Σ is an isomorphism of vector spaces. In particular, in the case of a Dirac operator, when a product structure near Y is assumed, we have:

$$\eth_{|U} = \sigma(\partial_t + \mathcal{B}),$$

i.e. $\Sigma = \sigma$ and $\mathcal{B}_t = \mathcal{B}$, which is a first order self-adjoint elliptic differential operator of $C^\infty(Y,E')$ independent of t. Also, $\mathcal{B}$ and σ anticommute, i.e. $\mathcal{B}\sigma = -\sigma\mathcal{B}$.

EXAMPLE 2.1.6. Let us consider the de Rham operator $d+\delta$ (Example 2.1.2) and $\omega \in \Omega^k(X)$ on a collar neighbourhood $U \cong [0,c) \times Y$. Since $\omega_{|U} = \omega_1 + dt \wedge \omega_2$

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[1] Definition 2.3, [**10**].

by (0.3.1), we have:

$$d^k(\omega_1 + dt \wedge \omega_2) = d_Y^k \omega_1 + dt \wedge (\partial_t \omega_1 - d_Y^{k-1}\omega_2) \quad \Longrightarrow \quad d^k = \begin{pmatrix} d_Y^k & 0 \\ \partial_t & -d_Y^{k-1} \end{pmatrix}$$

with respect to the decomposition

$$\Omega^k(X)_{|U} = \left(C^\infty([0,c)) \otimes \Omega^k(Y)\right) \oplus \left(dt \otimes C^\infty([0,c)) \otimes \Omega^{k-1}(Y)\right).$$

Analogously, since δ^k acts in a similar fashion on $\omega_{|U} \in \Omega^k(X)_{|U}$, we have (Lemma 3.1, [**38**]):

$$(d+\delta)_{|U} = \sigma \left[\partial_t + \overbrace{\sigma^{-1} \begin{pmatrix} d_Y + \delta_Y & 0 \\ 0 & -d_Y - \delta_Y \end{pmatrix}}^{\mathcal{B}} \right], \text{ where } \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $d_Y, \delta_Y : \Omega(Y) \to \Omega(Y)$. This can be obtained on a symbol level as follows. Let $\{e_0, \dots, e_{n-1}\}$ be an orthonormal basis for T^*X near Y with $e_0 = dt$, i.e. $T^*X \ni \xi = \sum_{j=0}^{n-1} \xi_j e_j$ and $\sigma^{d+\delta}(x,\xi) = i\mathbf{c}(\xi) = i\sum_{j=0}^{n-1} \xi_j \mathbf{c}(e_j)$. Then:

$$\begin{aligned}\sigma^{d+\delta}(0,y;D_t,\zeta) &= iD_t\mathbf{c}(e_0) + i\sum_{j=1}^{n-1} \zeta_j \mathbf{c}(e_j) = \partial_t \mathbf{c}(e_0) + i\sum_{j=1}^{n-1} \zeta_j \mathbf{c}(e_j) \\ &= \sigma\left(\partial_t + \sigma^{-1}\sigma^{d+\delta}(0,y;0,\zeta)\right) = \sigma\left(\partial_t + b(y,\zeta)\right),\end{aligned}$$

with $b(y,\zeta) := \sigma^{\mathcal{B}}(0,y;0,\zeta)$. In particular, $b(y,\zeta)$ has no purely immaginary eigenvalues (Lemma 1.9.4, [**23**]).

Remark 2.1.7. Since $\mathcal{B}$ is an elliptic self-adjoint operator on the closed manifold Y, it is well known that its spectrum is a discrete set of real eigenvalues with finite multiplicity and approaching $\pm\infty$. Let $V_\lambda \subset L^2(Y,E')$ denote the eigenspace of $\mathcal{B}$ associated to the eigenvalue λ. On the one hand, if $\Pi_\lambda : L^2(Y,E') \to L^2(Y,E')$ denotes the orthogonal projection into V_λ, we have that $\Pi_\lambda \in \Psi^{-\infty}(Y,E')$ (as it is finite rank). On the other, the orthogonal projection $\Pi_{\geq a}$ onto $V_{\geq a} := \bigoplus_{\lambda \geq a} V_\lambda$ is in $\Psi^0(Y,E')$, and so are $\Pi_{>a}$, $\Pi_{\leq a}$, and $\Pi_{<a}$ (Proposition 14.2, [**10**]).

2.2. Boundary value problems for Dirac operators

The following definitions and results are mainly taken from §18, [**10**], and §3, [**25**], unless otherwise stated.

Some topological invariants of closed manifolds can be obtained as the index of Dirac operators:

EXAMPLE 2.2.1. Let X be a closed $2n$-dimensional manifold. Then by the Atiyah-Singer Index Theorem $\chi(X) = \mathrm{ind}(d+\delta)^+$ (§1.5, [**23**]), where $(d+\delta)^+$ is the de Rham operator acting on $\Lambda^+(X) := \bigoplus_{j=0}^{n} \Lambda^{2j}(X)$, obtained from the $\mathbb{Z}_2$-grading $\Lambda(X) = \Lambda^+(X) \oplus \Lambda^-(X)$, with $\Lambda^-(X) := \bigoplus_{j=0}^{n-1} \Lambda^{2j+1}(X)$. A different $\mathbb{Z}_2$-grading of $\Lambda(X)$ yields a different invariant: in fact, if we consider the grading arising from the Hodge operator, then $\mathrm{ind}(d+\delta)^+ = \sigma(X)$, the topological signature of X (see Proposition 3.61, [**8**]).

When X has a non-empty boundary Y, a Dirac operator $\eth$ on X becomes Fredholm when suitable boundary conditions are imposed. We recall from §0.2 that X is considered embedded into a closed manifold $\widetilde{X}$ and that $E = \widetilde{E}_{|X}$, for $\widetilde{E} \to \widetilde{X}$ a Hermitian vector bundle. In analogy with the maps defined in §0.1 and §0.2, we consider $X_- := \overline{\widetilde{X} \setminus X}$, i.e. the closure of $\widetilde{X} \setminus X$, and $E_- := \widetilde{E}_{|X_-}$. Thus we have:

$$r^- : H^s(\widetilde{X}, \widetilde{E}) \to H^s(X_-, E_-), \qquad e^- : L^2(X_-, E_-) \to L^2(\widetilde{X}, \widetilde{E}),$$
$$\gamma^- : H^s(X_-, E_-) \to H^{s-\frac{1}{2}}(Y, E').$$

Let $X_+ := X$, $E_+ := E$, and $\gamma^+ := \gamma$. Recall that a pseudodifferential operator $D_\pm : C^\infty(X_\pm, E_\pm) \to C^\infty(X_\pm, E_\pm)$ is the truncation of a pseudodifferential operator $\widetilde{D}$ on $\widetilde{X}$ to $X_\pm$, i.e. $D_\pm := r^\pm \widetilde{D} e^\pm$.

DEFINITION 2.2.2. Let $N_\pm(y,\zeta)$ be the spaces of boundary values of the bounded solutions of $\sigma^{\eth}(y, 0, \zeta, D_t)z(t) = 0$ on $\mathbb{R}^\pm$, i.e.

$$N_\pm(y,\zeta) := \{z(0) |\ \sigma^{\eth}(0, y, D_t, \zeta)z(t) = 0,\ z(t) \to 0 \text{ as } t \to \pm\infty\}. \tag{2.2.1}$$

Then $\mathcal{P} \in \Psi^0(Y, E')$ is said to be a *well-posed boundary condition* for $\eth$ if the following two conditions are both fulfilled:

i) the extension $\mathcal{P}^s : H^s(Y, E') \to H^s(Y, E')$ has closed range for each $s \in \mathbb{R}$;
ii) for each (y, ζ) such that $\zeta \in T^*Y$ and $|\zeta| = 1$, $\sigma^{\mathcal{P}}(y,\zeta)$ maps $N_+(y,\zeta)$ injectively onto its range, i.e. $\sigma^{\mathcal{P}}(y,\zeta)_{|N_+(y,\zeta)} : N_+(y,\zeta) \to \mathrm{ran}(\sigma^{\mathcal{P}}(y,\zeta))$ is an isomorphism.

REMARK 2.2.3 (§3.6, [**6**]; Remark 18.2, [**10**]; §3, [**25**]). We note that in the literature if, for all $y \in Y$ and $|\zeta| = 1$, $\sigma^{\mathcal{P}}(y,\zeta)_{|N_+(y,\zeta)}$ is injective from $N_+(y,\zeta)$ to $\mathrm{ran}(\sigma^{\mathcal{P}}(y,\zeta))$, then the pair $(\eth, \mathcal{P}\gamma)$ is called *injectively elliptic*. If also $\sigma^{\mathcal{P}}(y,\zeta)_{|N_+(y,\zeta)}$ is surjective from $N_+(y,\zeta)$ to $\mathrm{ran}(\sigma^{\mathcal{P}}(y,\zeta))$ and there exists a sub-bundle $V \subset E'$ such that $\mathrm{ran}(\sigma^{\mathcal{P}}(y,\zeta)) = V_y$, then the pair $(\eth, \mathcal{P}\gamma)$ is called *surjectively elliptic*.

Boundary conditions that are both injective and surjectivene are called *local boundary conditions* ([**10**]) or properly elliptic ([**25**]). This implies $\dim N_{\pm}(y,\zeta) = \frac{1}{2}N$, as for relative and absolute boundary conditions $\mathcal{R}$ and $\mathcal{A}$.

Finally, although the terms *well-posed* and *elliptic* (as used in [**10**]) for these boundary conditions are used interchangeably, the latter can be confused with the standard terminology for 'elliptic' (which does not satisfy Definition 2.2.2). Hence, as in [**25**], we will adopt the term well-posed. A detailed explanation can be found in [**25**], §3.

DEFINITION 2.2.4. Let $\mathcal{P} \in \Psi^0(Y, E')$ satisfy Definition 2.2.2. Then a *realization* of $\eth$ is an unbounded operator

$$\eth_{\mathcal{P}} : \operatorname{dom}(\eth_{\mathcal{P}}) \to L^2(X,E), \qquad \operatorname{dom}(\eth_{\mathcal{P}}) = \{u \in H^1(X,E) \mid \mathcal{P}\gamma u = 0\}.$$

It is well known (Proposition 18.11, [**10**]) that $\mathcal{P} \in \Psi^0(Y, E')$ can be considered to be a non-trivial pseudodifferential projection, i.e. $\mathcal{P}^2 = \mathcal{P}$, and is orthogonal if $\mathcal{P}^* = \mathcal{P}$.

DEFINITION 2.2.5. Let $\mathcal{P} \in \Psi^0(Y, E')$ be a projection and set $p(y,\zeta) := \sigma^{\mathcal{P}}(y,\zeta)$. Then the *Grassmannian of pseudodifferential projections* with principal symbol p is the topological set:

$$\mathcal{G}_p := \{\mathcal{Q} \in \Psi^0(Y,E') |\ \mathcal{Q}^2 = \mathcal{Q} \text{ and } \sigma^{\mathcal{Q}} = p\}.$$

Let $\mathcal{P} \in \Psi^0(Y, E')$ be such that i) of Definition 2.2.2 is satisfied. Then the orthogonal L^2-projection $\mathcal{I}_{\mathcal{P}}$ onto $\operatorname{ran}(\mathcal{P}) \subset L^2(Y,E')$ is a pseudodifferential operator (Theorem 18.5, [**10**]).

PROPOSITION 2.2.6 ([**10**]). Let $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in \mathcal{G}_p$. Then:

i) $\mathcal{P}_2\mathcal{P}_1 : \operatorname{ran}(\mathcal{P}_1) \to \operatorname{ran}(\mathcal{P}_2)$ is Fredholm (note at page 119);
ii) $\operatorname{ind}(\mathcal{P}_1\mathcal{P}_2) + \operatorname{ind}(\mathcal{P}_2\mathcal{P}_3) = \operatorname{ind}(\mathcal{P}_1\mathcal{P}_3)$ (Proposition 15.15);
iii) $\operatorname{ind}(\mathcal{P}\mathcal{I}_{\mathcal{P}}) = \operatorname{ind}(\mathcal{I}_{\mathcal{P}}\mathcal{P}) = 0$ (Lemma 15.11);
iv) $\operatorname{ind}(\mathcal{P}_1\mathcal{P}_2) = \operatorname{ind}\left(\mathcal{P}_2^{\perp}\mathcal{P}_1^{\perp}\right)$, where $\mathcal{P}^{\perp} := I - \mathcal{P}$.

REMARK 2.2.7. Part iii) of Proposition 2.2.6 means that $\mathcal{P}$ and $\mathcal{I}_{\mathcal{P}}$ belong to the same connected component of $\mathcal{G}_p$.

The following theorem defines a special type of well-posed boundary conditions, which will be fundamental in the sequel. It needs the existance of an invertible Dirac operator $\widetilde{\eth}$ on the closed manifold $\widetilde{X}$, which can be obtained for instance by constructing the *closed double* of X. We refer to Theorem 9.1 of [**10**] for a description of such construction.

THEOREM 2.2.8 (Theorem 7.1, [**25**]). Let $\eth : C^\infty(X,E) \to C^\infty(X,E)$ be a Dirac operator and consider the spaces of *null-solutions* of $\eth$

$$Z^s_\pm := \{u \in H^s(X_\pm, E_\pm) \mid \eth u = 0 \text{ on } X_\pm\}$$

and *Cauchy data* of null-solutions of $\eth$, $N^s_\pm := \gamma^\pm Z^s_\pm \subset H^{s-\frac{1}{2}}(Y,E')$. Then:

i) the spaces $N^s_\pm$ are complementing subspaces of $H^{s-\frac{1}{2}}(Y,E')$, i.e.

$$H^{s-\frac{1}{2}}(Y,E') = N^s_+ \cup N^s_- \text{ and } N^s_+ \cap N^s_- = \{0\};$$

ii) there exist operators $\mathcal{K}^\pm := \pm r^\pm \widetilde{\eth}^{-1}\widetilde{\gamma}^*\sigma : H^{s-\frac{1}{2}}(Y,E') \to H^s(X_\pm, E_\pm)$, called *Poisson operators*, whose range is equal to $Z^s_\pm$ and $\mathcal{K}^\pm_{|N^s_\pm} : N^s_\pm \to Z^s_\pm$ are isomorphisms, i.e. Poisson operators are a left inverse of $\gamma^\pm$ on $Z^s_\pm$;

iii) there exist pseudodifferential projections

$$\mathcal{C}^\pm := \gamma^\pm\mathcal{K}^\pm : H^{s-\frac{1}{2}}(Y,E') \to H^{s-\frac{1}{2}}(Y,E'),$$

called *Calderón projectors*, whose range is equal to $N^s_\pm$ (along $N^s_\mp$), i.e. $\mathcal{C}^+ + \mathcal{C}^- = I$ and $\mathcal{C}^\pm\mathcal{C}^\mp = 0$.

REMARK 2.2.9. $\mathcal{C}^\pm$ are projections because $\mathcal{K}^\pm$ is a left inverse for $\gamma^\pm$ on $Z^s_\pm$ ([**26**]), i.e. $(\mathcal{C}^\pm)^2 = \gamma^\pm\mathcal{K}^\pm\gamma^\pm\mathcal{K}^\pm = \gamma^\pm\mathcal{K}^\pm = \mathcal{C}^\pm$. Also, although $\mathcal{C}^\pm$ are not orthogonal a priori, they can be considered to be so by iii) of Proposition 2.2.6.

Finally, by Unique Continuation Property (Theorem 2.1.3) there are no non-trivial solutions of $\eth s = 0$ with support all contained in X (Remark 12.2, [**10**]).

REMARK 2.2.10. Since the symbols $\sigma^{\mathcal{C}^\pm}(y,\zeta)$ are the orthogonal projections onto $N_\pm(y,\zeta)$, i.e. (2.2.1), $N_+(y,\zeta)$ and $N_-(y,\zeta)$ are orthogonal complements and $\mathrm{ran}\sigma^{\mathcal{C}^+} \cong \mathrm{ran}\sigma^{\mathcal{P}}$ for every well-posed boundary condition $\mathcal{P}$ by part ii) of Definition 2.2.2.

Moreover, $N_\pm(y,\zeta)$ correspond to the generalised eigenspaces associated with the positive, respectively negative, eigenvalues of $\sigma^{\mathcal{B}}(y,\zeta)$, and thence $\sigma^{\mathcal{C}^\pm}(y,\zeta)$ coincide with the principal symbol of the spectral projections of Remark 2.1.7, i.e. $\sigma^{\mathcal{C}^+} = \sigma^{\Pi_{\geq 0}} = \sigma^{\Pi_{>0}}$ and $\sigma^{\mathcal{C}^-} = \sigma^{\Pi_{\leq 0}} = \sigma^{\Pi_{<0}}$. Since $\sigma^{\Pi_{\geq 0}} = \sigma^{\Pi_{\geq a}}$ for all $a \in \mathbb{R}$, this shows that the symbols are independent of a.

Since $\Pi_{\geq a}, \Pi_{>a} \in \mathcal{G}_{\sigma^{\mathcal{C}^+}}$ and $\Pi_{\leq a}, \Pi_{>a} \in \mathcal{G}_{\sigma^{\mathcal{C}^-}}$, then the differences are compact operators, i.e. $\mathcal{C}^+ - \Pi_{\geq a} \in \Psi^{-1}(Y,E')$ and $\mathcal{C}^- - \Pi_{\leq a} \in \Psi^{-1}(Y,E')$, $\forall a \in \mathbb{R}$. In the case of compact manifolds, this can be improved:

PROPOSITION 2.2.11 (Proposition 2.2, [**71**]). If X is compact with product metric near Y, then $\mathcal{C}^+ - \Pi_{\geq a} \in \Psi^{-\infty}(Y, E')$ and $\mathcal{C}^- - \Pi_{\leq a} \in \Psi^{-\infty}(Y, E')$.

DEFINITION 2.2.12. Let $\mathcal{P} \in \Psi^0(Y, E')$ be a well-posed boundary condition. Then the operator $\mathcal{P}\mathcal{C}^+ : \operatorname{ran}(\mathcal{C}^+) \to \operatorname{ran}(\mathcal{P})$ is called the *boundary integral* associated to $\eth_{\mathcal{R}}$.

THEOREM 2.2.13. Let $\mathcal{P} \in \Psi^0(Y, E')$ be an well-posed boundary condition and let $\mathcal{I}_{\mathcal{P}\mathcal{C}^+}$ denote the orthogonal projections of $L^2(Y, E')$ onto $\operatorname{ran}(\mathcal{P}\mathcal{C}^+)$ and $\mathcal{I}_{\mathcal{C}^+\mathcal{P}^*}$ denote the one onto $\operatorname{ran}(\mathcal{C}^+\mathcal{P}^*)$. Then:

i) $\mathcal{I}_{\mathcal{P}\mathcal{C}^+} \in \mathcal{G}_{\sigma^{\mathcal{P}}}$ and $\mathcal{I}_{\mathcal{C}^+\mathcal{R}^*} \in \mathcal{G}_{\sigma^{\mathcal{C}^+}}$;

ii) $\mathcal{P}\mathcal{C}^+ : \operatorname{ran}(\mathcal{C}^+) \to \operatorname{ran}(\mathcal{P})$ is Fredholm and

$$\operatorname{ind}(\mathcal{P}\mathcal{C}^+) = \operatorname{ind}(\mathcal{I}_{\mathcal{C}^+\mathcal{P}^*}\mathcal{C}^+) - \operatorname{ind}(\mathcal{I}_{\mathcal{P}\mathcal{C}^+}\mathcal{I}_{\mathcal{P}}); \tag{2.2.2}$$

iii) $\eth_{\mathcal{P}}$ is Fredholm operator and:

$$\operatorname{ind}(\eth_{\mathcal{P}}) = \operatorname{ind}(\mathcal{P}\mathcal{C}^+). \tag{2.2.3}$$

2.3. The additivity of the index on a partitioned closed manifold

Let $\eth_i : C^\infty(X_i, E_i) \to C^\infty(X_i, E_i)$, $i = 1, 2$, be two Dirac operators over X_i, such that $\partial X_1 = \partial X_2 = Y$.

DEFINITION 2.3.1 (§23, [**10**]). $\eth_1$ and $\eth_2$ are *consistent* if in a collar neighbourhood of $\partial X_1 = \partial X_2 = Y$ they can be represented in the following form:

$$\eth_1 = \sigma\left(\partial_t + \mathcal{B}\right) \qquad \eth_2 = \sigma^{-1}\left(\partial_v - \sigma\mathcal{B}\sigma^{-1}\right), \qquad t = -v.$$

REMARK 2.3.2. For example, if $\eth$ is a Dirac operator on a closed manifold X that we partition with respect to a 1-codimensional submanifold Y into $X_1 \cup_Y X_2$, then $\eth$ restricts to two consistent Dirac operators $\eth_i := \eth_{|X_i}$.

In fact, $\sigma\mathcal{B}\sigma^{-1}$ corresponds to the boundary Dirac operator when Y has opposite orientation and $\eth_2$ is formally equal to $\eth_1^*$ close to Y. Thus, via gluing (Chapter 9, [**10**]), we obtain the Dirac operator $\eth$ on the manifold $X = X_1 \cup_Y X_2$, which restricts to $\eth_i$ over X_i, $i = 1, 2$.

Formula (2.2.3) shows that for the realization of a Dirac operator on a manifold X, the data related to the index are encoded in the boundary. Therefore, when X is closed, one can obtain the value of the index of an associated Dirac operator via a choice of 1-codimensional splitting embedded submanifold. In other words, for a closed submanifold $Y \hookrightarrow X$ we obtain a splitting $X = X_1 \cup_Y X_2$, where $X_i \subseteq X$,

$i = 1, 2$, has Y as a common boundary (with reverse orientation in one case). Let $\eth : C^\infty(X, E) \to C^\infty(X, E)$ be a Dirac operator on X, restricting to $\eth_i$ on X_i, $i = 1, 2$. If we assume a bicollar neighbourhood for Y, with product structure, we have:

THEOREM 2.3.3 (24.1, [**10**]). Let $\mathcal{C}_i := \mathcal{C}_i^+$ be the Calderón projectors associated to $\eth_i$, $i = 1, 2$. Then:

$$\operatorname{ind}(\eth) = \operatorname{ind}\left(\mathcal{C}_2^\perp \mathcal{C}_1\right). \tag{2.3.1}$$

Clearly, formula (2.3.1) has a obvious similarity with what stated in Remark 1.4.32. It yields the following:

COROLLARY 2.3.4. Let $\mathcal{P}, \mathcal{Q} \in \mathcal{G}_{\sigma \mathcal{C}_1}$. Then:

$$\operatorname{ind}(\eth) - \operatorname{ind}(\mathcal{Q}\mathcal{P}) = \operatorname{ind}(\eth_{1,\mathcal{P}}) + \operatorname{ind}\left(\eth_{2,\mathcal{Q}^\perp}\right) = \operatorname{ind}(\mathcal{P}\mathcal{C}_1) + \operatorname{ind}\left(\mathcal{Q}^\perp \mathcal{C}_2\right). \tag{2.3.2}$$

In general, $\operatorname{ind}(\mathcal{Q}\mathcal{P}) \neq 0$ and we do not have strict additivity. However, by Proposition 2.2.6, it can be possible to change the boundary conditions $\mathcal{P}$ and $\mathcal{Q}$ so that the extra term will vanish.

EXAMPLE 2.3.5. Let $\eth_\Pi$ denote the realization of a Dirac operator $\eth$ over X with *APS boundary conditions*, i.e. with $\Pi := \Pi_{\geq 0}$ from Remark 2.1.7. Let ω denote the index density, $\eta(\mathcal{B}) := \eta(0, \mathcal{B})$ be the *eta invariant* of $\mathcal{B}$, and set $h(\mathcal{B}) := \dim\ker(\mathcal{B})$. Then the *Atiyah-Patodi-Singer Index Theorem* ([**4**]) shows that:

$$\operatorname{ind}(\eth_\Pi) = \int_X \omega - \frac{\eta(\mathcal{B}) + h(\mathcal{A})}{2}. \tag{2.3.3}$$

Let $\eth_i : C^\infty(X_i, E_i) \to C^\infty(X_i, E_i)$ be two consistent Dirac operators over X_i, such that $\partial X_1 = \partial X_2 = Y$, and set Π_i for the pseudodifferential projection $\Pi_{\geq 0}$ for $\eth_i$. Then (Proposition 23.2, [**10**]):

$$\operatorname{ind}(\eth) = \operatorname{ind}(\eth_{1,\Pi}) + \operatorname{ind}(\eth_{2,\Pi}) + h(\mathcal{B})$$

where we set $\eth_{i,\Pi} := \eth_{i,\Pi_i}$. Hence additivity holds if and only if $h(\mathcal{B}) = 0$. By ii) of Proposition 2.2.6 and equality (2.2.3), for another $\mathcal{P} \in \mathcal{G}_{\sigma \mathcal{C}^+}$ we have the *Agranovic-Dynin formula*:

$$\operatorname{ind}(\eth_\mathcal{P}) = \operatorname{ind}(\eth_\Pi) + \operatorname{ind}(\mathcal{P}\Pi). \tag{2.3.4}$$

If $\ker(\mathcal{B}) \neq \{0\}$, then there always exists a unitary involution $\tau : \ker(\mathcal{B}) \to \ker(\mathcal{B})$, determined by the Dirac operator $\eth$ and anticommuting with σ, i.e. $\sigma\tau = -\tau\sigma$ (Proposition 4.26, [**56**]). The ± 1-eigenspaces of τ, $L_\pm := \ker(\tau \mp \mathrm{id})$, are Lagrangian subspaces[2] of $\ker(\mathcal{B})$, i.e. $L_\pm = \sigma L_\mp$ and $\ker(\mathcal{B}) = L_+ \oplus L_-$. In particular, $h(\mathcal{B}) = \dim\ker(\mathcal{B}) \in 2\mathbb{N}$.

Let $\Theta_\pm$ the orthogonal projections of $L^2(Y, E')$ onto $L_\pm$. Hence, $\Theta_\pm$ are finite rank projections and define the *generalized* APS boundary conditions:

$$\mathcal{P}^\pm_> := \Pi_{>0} + \Theta_\pm \in \mathcal{G}_{\sigma c^+} \qquad \text{and} \qquad \mathcal{P}^\pm_< := \Pi_{<0} + \Theta_\pm \in \mathcal{G}_{\sigma c^-}.$$

In particular, they are well-posed and

$$\begin{aligned}
&\operatorname{ind}(\mathcal{P}^\pm_>\Pi : \operatorname{ran}(\Pi) \to \operatorname{ran}(\mathcal{P}^\pm_>) = \\
&\quad = \dim(\operatorname{ran}(\Pi) \cap \operatorname{ran}(\mathcal{P}^\pm_>)^\perp) - \dim(\operatorname{ran}(\Pi)^\perp \cap \operatorname{ran}(\mathcal{P}^\pm_>)) \\
&\quad = \dim(\operatorname{ran}(\Pi) \cap \operatorname{ran}(\mathcal{P}^\mp_<)) - \underbrace{\dim(\operatorname{ran}(\Pi_{<0}) \cap \operatorname{ran}(\mathcal{P}^\pm_>))}_{=0} \\
&\quad = \dim(L_\mp) = \dim(L_\pm) = \frac{1}{2}\dim\ker(\mathcal{B}) = \frac{1}{2}h(\mathcal{B}).
\end{aligned}$$

Analogously, $\operatorname{ind}(\mathcal{P}^\pm_<\Pi_{\le 0}) = \frac{1}{2}h(\mathcal{B})$ and, in conclusion, if $\eth_{i,\pm}$ denote the realizations of $\eth_i$ with respect to either one of the generalized APS boundary conditions, we have by (2.3.4)

$$\operatorname{ind}(\eth_{i,\pm}) = \operatorname{ind}(\eth_{i,\Pi}) + \operatorname{ind}(\mathcal{P}^\pm_>\Pi) = \int_{X_i} \omega_i - \frac{1}{2}\eta(\mathcal{B})$$

and $\operatorname{ind}(\eth) = \operatorname{ind}(\eth_1^\pm) + \operatorname{ind}(\eth_2^\pm)$.

Remark 2.3.6. When $\eth$ is the signature operator, the above example is used to show that the topological signature of a manifold can be realised as the trace character of a logTQFT. See **[72]** for further details on this.

2.4. The additivity of the index for manifolds with boundary

Example 2.4.1. We continue Example 2.3.5 by considering this time two manifolds X_i, $i = 1, 2$, such that $\partial X_i = Y_{i-1}^- \sqcup Y_i$ and at least one between Y_0 and Y_2 is non-empty. If the two Dirac operators $\eth_i : C^\infty(X_i, E_i) \to C^\infty(X_i, E_i)$ are consistent in a collar neighbourhood of Y_1, then we can glue them together into a Dirac operator $\eth : C^\infty(X, E) \to C^\infty(X, E)$ over $X = X_1 \cup_{Y_1} X_2$, which has non-empty boundary $\partial X = Y_0^- \sqcup Y_2$. If $\mathcal{B}_i := \mathcal{B}_{|Y_i}$, then $\mathcal{B}_{|\partial X_1} = \sigma\mathcal{B}_0\sigma^{-1} \oplus \mathcal{B}_1$ and $\mathcal{B}_{|\partial X_2} = \sigma\mathcal{B}_1\sigma^{-1} \oplus \mathcal{B}_2$. Therefore, it si well known that:

$$\eta(\mathcal{B}_{|\partial X_1}) = -\eta(\mathcal{B}_0) + \eta(\mathcal{B}_1) \quad \text{and} \quad \eta(\mathcal{B}_{|\partial X_2}) = -\eta(\mathcal{B}_1) + \eta(\mathcal{B}_2)$$

[2]Definition: L_+ is the space of all limiting values of L^2-extended sections u of E satisfying $\eth_i u = 0$.

and by the Atiyah-Patodi-Singer Index Theorem:

$$\begin{aligned}\operatorname{ind}(\eth_{1,\Pi})+\operatorname{ind}(\eth_{2,\Pi}) &= \int_{X_1}\omega_1-\frac{\eta(\sigma\mathcal{B}_0\sigma^{-1}\oplus\mathcal{B}_1)+h(\sigma\mathcal{B}_0\sigma^{-1}\oplus\mathcal{B}_1)}{2}\\ &+\int_{X_2}\omega_2-\frac{\eta(\sigma\mathcal{B}_1\sigma^{-1}\oplus\mathcal{B}_2)+h(\sigma\mathcal{B}_1\sigma^{-1}\oplus\mathcal{B}_2)}{2}\\ &=\int_{X_1}\omega_1-\frac{-\eta(\mathcal{B}_0)+h(\mathcal{B}_0)}{2}-\frac{\eta(\mathcal{B}_1)+h(\mathcal{B}_1)}{2}\\ &+\int_{X_2}\omega_2-\frac{-\eta(\mathcal{B}_1)+h(\mathcal{B}_1)}{2}-\frac{\eta(\mathcal{B}_2)+h(\mathcal{B}_2)}{2}\\ &=\int_X\omega-\frac{-\eta(\mathcal{B}_0)+h(\mathcal{B}_0)}{2}-\frac{\eta(\mathcal{B}_2)+h(\mathcal{B}_2)}{2}-h(\mathcal{B}_1)\\ &=\operatorname{ind}(\eth_\Pi)+\dim\ker(\mathcal{B}_1),\end{aligned}$$

Let $\Theta_1^{\pm}: L^2(Y_1,E_1')\to L^2(Y_1,E_1')$ be the orthogonal projection onto the Lagrangian subspaces $L_\pm$ of $\ker(\mathcal{B}_1)$, and Π_i denote the APS projection corresponding to $\mathcal{B}_1$. Then suitable conditions on X_1 and X_2 are, respectively,

$$\mathcal{P}_1^{\pm}=\begin{pmatrix}\Pi_{0,\le 0} & 0\\ 0 & \Pi_{1,>0}+\Theta_1^{\pm}\end{pmatrix}\quad\text{and}\quad \mathcal{P}_2^{\pm}=\begin{pmatrix}\Pi_{1,<0}+\Theta_1^{\pm} & 0\\ 0 & \Pi_{2,\ge 0}\end{pmatrix}.$$

In fact, if we denote $\eth_{\mathcal{P}_i^{\pm}}$ the realization of $\eth_i$ with such generalized APS conditions, we obtain

$$\begin{aligned}\operatorname{ind}(\eth_{\mathcal{P}_1^{\pm}}) &= \int_{X_1}\omega_1-\frac{-\eta(\mathcal{B}_0)+\dim\ker(\mathcal{B}_0)}{2}-\frac{\eta(\mathcal{B}_1)}{2},\\ \operatorname{ind}(\eth_{\mathcal{P}_2^{\pm}}) &= \int_{X_2}\omega_2-\frac{\eta(\mathcal{B}_2)+\dim\ker(\mathcal{B}_2)}{2}+\frac{\eta(\mathcal{B}_1)}{2},\end{aligned}$$

and thus:

$$\operatorname{ind}(\eth_{\mathcal{P}_1^{\pm}})+\operatorname{ind}(\eth_{\mathcal{P}_2^{\pm}})=\int_X\omega-\frac{-\eta(\mathcal{B}_0)+\eta(\mathcal{B}_2)}{2}-\frac{h(\mathcal{B}_0)+h(\mathcal{B}_2)}{2}=\operatorname{ind}(\eth_\Pi).$$

This example is just an instance of the general fact that formula (2.3.2) holds also when gluing is performed with respect to a proper subset of the connected components of the boundary (Remark 8.20, [**7**]). In that case, one imposes generic well-posed boundary conditions of the form:

$$\mathcal{P}=\begin{pmatrix}\mathcal{P}_1 & 0\\ 0 & \mathcal{P}_2\end{pmatrix}$$

on the remaining boundary components.

Here, we will prove this additive formula but from the point of view of the Calderón projectors and boundary integrals. It is clear that this is just an equivalent formulation of what has just been stated.

For a general approach to gluing, let us consider a manifold X with boundary $Y := \bigsqcup_{i=0}^{k} Y_i$, for $k \in \mathbb{N}$. Hence $E' = \bigoplus_{i=0}^{k} E'_i$, with $E'_i := E_{|Y_i}$, and $H^s(Y, E') = \bigoplus_{i=0}^{k} H^s(Y_i, E'_i)$. For $U_i := [0, c_i) \times Y_i$ a collar neighbourhood of Y_i, the set $U = [0, \max_i c_i] \times Y \supseteq \bigsqcup_{i=0}^{k} U_i$ is a collar neighbourhood of Y. Then the product structure near the boundary implies $\eth_{|U_i} = \sigma_i(\partial_{u_i} + \mathcal{B}_i)$, with $\sigma_i : E_{|U_i} \to E_{|U_i}$ the Clifford multiplication by unit inward normal vector to Y_i and $\mathcal{B}_i := \mathcal{B}_{|Y_i}$ the restriction to Y_i.

First of all, Theorem 11.4 and Corollary 11.8 of [**10**] can be reformulated for every 1-codimensional embedded submanifold in X:

THEOREM 2.4.2. $\forall s > \frac{1}{2}$ and $\forall i \in \{0, \ldots, k\}$, the restriction to the boundary component Y_i defines continuous and uniformly bounded trace maps

$$\gamma_i : H^s(X, E) \to H^{s-\frac{1}{2}}(Y_i, E'_i) \qquad \text{and} \qquad \widetilde{\gamma}_i : H^s(\widetilde{X}, \widetilde{E}) \to H^{s-\frac{1}{2}}(Y_i, E'_i).$$

In particular, $\widetilde{\gamma}_i$ are adjointable.

In this context, Green's formula becomes, for $s_1, s_2 \in C^\infty(X, E)$:

$$\begin{aligned}\langle \eth s_1, s_2 \rangle_X - \langle s_1, \eth s_2 \rangle_X &= -\int_Y (\sigma\gamma u, \gamma v)_y v(y) dy \\ &= -\sum_{i=0}^{k} \int_{Y_i} (\sigma_i \gamma_i u, \gamma_i v)_y v(y) dy = -\sum_{i=0}^{k} \langle \sigma_i \gamma_i u, \gamma_i v \rangle_{Y_i},\end{aligned}$$

since the restrictions γ and $\widetilde{\gamma}$ can be represented as column vectors, and σ, $\eth_{|U}$ and $\Pi_{\geq}$ as diagonal matrices:

$$\gamma = \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_k \end{pmatrix} : H^s(X, E) \to \bigoplus_{i=0}^{k} H^{s-\frac{1}{2}}(Y_i, E'_i)$$

$$\sigma = \begin{pmatrix} \sigma_0 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} : \bigoplus_{i=0}^{k} E'_i \to \bigoplus_{i=0}^{k} E'_i, \qquad \Pi_{\geq 0} = \begin{pmatrix} \Pi_{0,\geq 0} & & \\ & \ddots & \\ & & \Pi_{k,\geq 0} \end{pmatrix}$$

$$\eth_{|U} = \begin{pmatrix} \eth_{|U_0} & & \\ & \ddots & \\ & & \eth_{|U_k} \end{pmatrix} : \bigoplus_{i=0}^{k} C^\infty(U_i, E_{|U_i}) \to \bigoplus_{i=0}^{k} C^\infty(U_i, E_{|U_i})$$

Poisson operator and Calderón projector

$$\mathcal{K} : \bigoplus_{i=0}^{k} H^{s-\frac{1}{2}}(Y_i, E_i') \to H^s(X, E) \qquad \mathcal{K} := \mathcal{K}^+ = r\widetilde{\eth}^{-1}\widetilde{\gamma}^*\sigma$$

$$\mathcal{C} : \bigoplus_{i=0}^{k} H^{s-\frac{1}{2}}(Y_i, E_i') \to \bigoplus_{i=0}^{k} H^{s-\frac{1}{2}}(Y_i, E_i') \qquad \mathcal{C} := \mathcal{C}^+ = \gamma\mathcal{K}$$

can be represented in the following way:

$$\mathcal{K} = r\widetilde{\eth}^{-1}\left(\widetilde{\gamma}_0^*, \dots, \widetilde{\gamma}_k^*\right)\begin{pmatrix} \sigma_0 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{pmatrix} = (r\widetilde{\eth}^{-1}\widetilde{\gamma}_0^*\sigma_0, \dots, r\widetilde{\eth}^{-1}\widetilde{\gamma}_k^*\sigma_k) = (\mathcal{K}_0, \dots, \mathcal{K}_k),$$

$$\mathcal{C} = \left(\gamma_i r\widetilde{\eth}^{-1}\widetilde{\gamma}_j^*\sigma_j\right)_{i,j=0}^{k} = (\gamma_i\mathcal{K}_j)_{i,j=0}^{k} = (\mathcal{C}_{i,j})_{i,j=0}^{k},$$

since $\widetilde{\gamma}^* = (\widetilde{\gamma}_0^*, \dots, \widetilde{\gamma}_k^*)$,

REMARK 2.4.3. We have already seen in Remark 2.2.9 that $\mathcal{C}^2 = \mathcal{C}$ because $\mathcal{K}$ is a left inverse of γ. Similarly, in this case we have $\sum_{i=0}^{k}\mathcal{K}_i\gamma_i u = u$ for $u \in Z^1$, and $\mathcal{C}^2 = \mathcal{C}$ as a consequence. In fact, let $u \in Z^1$, $w \in L^2(X, E)$ and $v = (\widetilde{\eth}^{-1})^* ew$. Hence $v \in H^1(\widetilde{X}, \widetilde{E})$ and by Green's Formula:

$$\begin{aligned}
-\langle u, w\rangle &= \langle \eth u, rv\rangle_X - \langle u, r\widetilde{\eth}^* v\rangle_X = \langle \eth u, rv\rangle_X - \langle u, \eth rv\rangle_X \\
&= -\sum_{i=0}^{k}\langle \sigma_i\gamma_i u, \gamma_i rv\rangle_{Y_i} = -\sum_{i=0}^{k}\langle \sigma_i\gamma_i u, \widetilde{\gamma}_i v\rangle_{Y_i} \\
&= -\sum_{i=0}^{k}\langle \widetilde{\gamma}_i^*\sigma_i\gamma_i u, (\widetilde{\eth}^{-1})^* ew\rangle_{\widetilde{X}} = -\sum_{i=0}^{k}\langle r\widetilde{\eth}^{-1}\widetilde{\gamma}_i^*\sigma_i\gamma_i u, w\rangle_X \\
&= -\langle\sum_{i=0}^{k}\mathcal{K}_i\gamma_i u, w\rangle_X \quad \Longrightarrow \quad \sum_{i=0}^{k}\mathcal{K}_i\gamma_i u = u \in Z^1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{C}^2 &= \left(\sum_{l=0}^{k}\mathcal{C}_{i,l}\mathcal{C}_{l,j}\right)_{i,j=0}^{k} = \left(\sum_{l=0}^{k}\gamma_i\mathcal{K}_l\gamma_l\mathcal{K}_j\right)_{i,j=0}^{k} \\
&= \left(\gamma_i(\sum_{l=0}^{k}\mathcal{K}_l\gamma_l)\mathcal{K}_j\right)_{i,j=0}^{k} = (\gamma_i\mathcal{K}_j)_{i,j=0}^{k} = \mathcal{C}.
\end{aligned}$$

EXAMPLE 2.4.4. If $Y = Y_0 \sqcup Y_1$:

$$\mathcal{C} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix} r\widetilde{\eth}^{-1}\left(\widetilde{\gamma}_0^*, \widetilde{\gamma}_1^*\right)\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_0 r\widetilde{\eth}^{-1}\widetilde{\gamma}_0^*\sigma_0 & \gamma_0 r\widetilde{\eth}^{-1}\widetilde{\gamma}_1^*\sigma_1 \\ \gamma_1 r\widetilde{\eth}^{-1}\widetilde{\gamma}_0^*\sigma_0 & \gamma_1 r\widetilde{\eth}^{-1}\widetilde{\gamma}_1^*\sigma_1 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{0,0} & \mathcal{C}_{0,1} \\ \mathcal{C}_{1,0} & \mathcal{C}_{1,1} \end{pmatrix}.$$

LEMMA 2.4.5. Let $u \in C^\infty(X,E)$ such that $\eth u = 0$ and assume that there exists $i \in \{0,\dots,k\}$ such that $\gamma_i u = 0$. Then $u = 0$ on X and therefore $\forall i \in \{0,\dots,k\}$ $\gamma_i u = 0$.

PROOF. The argument in Lemma 2.3, [**71**], works equally well for manifolds with boundary with multiple connected components. Fix $i \in \{0,\dots,k\}$ and let $\eth_i$ denote $\eth$ restricted to $C_i^\infty(X,E) := \{u \in C^\infty(X,E) \mid \gamma_i u = 0\}$. By the product structure near Y_i, $u(t,y) = \sum_\lambda u_\lambda(t)v_\lambda(y)$ in U_i. Since $\eth_i u = 0$, $u_\lambda(t) = e^{-\lambda t}u_\lambda(0)$. Hence $u = 0$ on U_i because $u_\lambda(0) = 0$. By Unique Continuation Property the claim follows.

□

COROLLARY 2.4.6. Let $u, v \in C^\infty(X,E)$ such that $\eth u = \eth v = 0$ and assume that $\exists i \in \{0,\dots,k\}$ such that $\gamma_i u = \gamma_i v$. Then $u = v$ on X and therefore $\gamma_j u = \gamma_j v$ $\forall j \in \{0,\dots,k\}$.

COROLLARY 2.4.7. $\gamma_i : \ker\eth \to \gamma_i \ker\eth$ is bijective.

PROOF. Direct consequence of Corollary 2.4.6.

□

PROPOSITION 2.4.8. $\mathcal{C}_{i,j} = \gamma_i r\widetilde{\eth}^{-1}\widetilde{\gamma}_j^*\sigma_j$ is smoothing for $i \neq j$.

PROOF. Let $\xi \in C^\infty(Y_j, E_j')$, $i \in \{0,\dots,k\}$. By pseudolocality of $\widetilde{\eth}^{-1}$ ([**10**]), the singular support of $\widetilde{\gamma}_j^*\sigma_j\xi$ is contained in Y_j; hence, $r\widetilde{\eth}^{-1}\widetilde{\gamma}_j^*\sigma_j\xi$ is C^∞ outside Y_j, which implies that $\mathcal{C}_{i,j} = \gamma_i r\widetilde{\eth}^{-1}\widetilde{\gamma}_j^*\sigma_j$ is smoothing for $i \neq j$.

□

COROLLARY 2.4.9. For each $i \in \{0,\dots,k\}$, $\mathcal{C}_{i,i}$ is a projection modulo smoothing operators, i.e. $\mathcal{C}_{i,i}^2 - \mathcal{C}_{i,i} \in \Psi^{-\infty}(Y_i, E_i')$.

PROOF. Since $\mathcal{C}^2 = \mathcal{C}$ (Remark 2.4.3), we have:

$$\mathcal{C}_{i,i} = (\mathcal{C}^2)_{i,i} = \sum_{j=0}^{k} \mathcal{C}_{i,j}\mathcal{C}_{j,i} = \mathcal{C}_{i,i}^2 + \sum_{j\neq i} \mathcal{C}_{i,j}\mathcal{C}_{j,i}.$$

Then the claim follows by Proposition 2.4.8.

□

THEOREM 2.4.10. If X is compact with product metric near the boundary, then $\Pi_{i,\geq 0} - \mathcal{C}_{i,i} \in \Psi^{-\infty}(Y_i, E_i') \; \forall i \in \{0, \dots, k\}$.

PROOF. For all $i \in \{0, \dots, 1\}$, $\mathcal{C}_{i,i}$ is a pseudodifferential operator with the same principal symbol of $\Pi_{i,\geq 0}$, hence $\Pi_{i,\geq 0} - \mathcal{C}_{i,i} \in \Psi^{-1}(Y_i, E_i')$ in general. In particular, since X is compact with product metric, by Proposition 2.2.11 we have $\Pi_{\geq 0} - \mathcal{C} \in \Psi^{-\infty}(Y, E')$; thus, the diagonal components of $\mathcal{C}$ differ from those of $\Pi_{\geq 0}$ by a smoothing operator.

□

COROLLARY 2.4.11. Let Y_i^- denote Y_i with opposite orientation and $\widetilde{\mathcal{C}}_{i,i}$ be the Calderón projector defined for Y_i^-. If $\Pi_{\geq 0} - \mathcal{C} \in \Psi^{-\infty}(Y, E')$, then

$$\mathcal{C}_{i,i} + \widetilde{\mathcal{C}}_{i,i} - I \in \Psi^{-\infty}(Y_i, E_i').$$

PROOF. Let $\Pi^-_{\geq 0}$ denote the projection onto the non-negative eigenspace of $\mathcal{B}$ when the orientation of Y_i is reversed. Then $\Pi^-_{i,\geq 0} = \Pi_{i,\leq 0}$ and, since $\Pi_{i,\geq 0} - \mathcal{C}_{i,i} \in \Psi^{-\infty}(Y_i, E_i')$, we have that $\Pi_{i,\leq 0} - \widetilde{\mathcal{C}}_{i,i} \in \Psi^{-\infty}(Y_i, E_i')$. Therefore,

$$\Psi^{-\infty}(Y_i, E_i') \ni \mathcal{C}_{i,i} + \widetilde{\mathcal{C}}_{i,i} - \Pi_{i,\geq 0} - \Pi_{i,\leq 0} = \mathcal{C}_{i,i} + \mathcal{C}^-_{i,i} - I - \Pi_{i,0}.$$

Hence the statement, since $\Pi_{i,0}$ is finite rank.

□

LEMMA 2.4.12. Let $\mathcal{C} := \mathcal{C}^+$ be the Calderón projector for a Dirac operator $\eth : C^\infty(X, E) \to C^\infty(X, E)$ and assume $Y := \bigsqcup_{i=0}^k Y_i$, for $k \in \mathbb{N}$. Then there exists an orthogonal projection $\mathcal{I}_{\mathcal{C}}$ onto $\operatorname{ran}(\mathcal{C})$ that is diagonal with respect to the Boundary decomposition and such that $\mathcal{I}_{\mathcal{C}}$ and $\mathcal{C}$ belong to the same connected component of $\mathcal{G}_{\sigma^{\mathcal{C}}}$.

PROOF. Without loss of generality, by ii) of Proposition 2.2.6, we can assume $\mathcal{C} = \mathcal{C}^*$. Hence, $\mathcal{C}_{i,i} = \mathcal{C}^*_{i,i}$ and $\mathcal{C}_{i,j} = \mathcal{C}^*_{j,i}$, $i = 0, 1$. Consider the operator:

$$\widetilde{\mathcal{C}} = \bigoplus_{i=0}^k \mathcal{C}_{i,i} = \begin{pmatrix} \mathcal{C}_{0,0} & & 0 \\ & \ddots & \\ 0 & & \mathcal{C}_{k,k} \end{pmatrix}.$$

Since $\mathcal{C}_{i,j}$ are smoothing, $\widetilde{\mathcal{C}}$ it is a smooth perturbation of $\mathcal{C}$ and thus is a well-posed boundary condition. As such, its range is closed for every s. In particular, its range is $\operatorname{ran}(\widetilde{\mathcal{C}}) = \bigoplus_{i=0}^k \operatorname{ran}(\mathcal{C}_{i,i})$, from which we conclude that $\mathcal{C}_{i,i} \in \Psi^0(Y_i, E_i')$ has closed range. Let $\mathcal{I}_{i,i}$ denote the L^2-orthogonal projection onto $\operatorname{ran}(\mathcal{C}_{i,i})$. By

Theorem 18.5, [**10**], $\mathcal{I}_{i,i} \in \Psi^0(Y_i, E_i')$ and the operator $\mathcal{I}_{\mathcal{C}} := \bigoplus_{i=0}^k \mathcal{I}_{i,i} \in \Psi^0(Y, E')$ is a projection, and by ii) of Proposition 2.2.6, $\operatorname{ind}(\eth_{\widetilde{\mathcal{C}}}) = \operatorname{ind}(\eth_{\mathcal{I}_{\mathcal{C}}})$.

Finally, as $\widetilde{\mathcal{C}} = \mathcal{C} + \mathcal{S}$ for $\mathcal{S} \in \Psi^{-\infty}(Y, E')$, the Fredholm operator $\widetilde{\mathcal{C}}\mathcal{C}$ is a compact perturbation of $\mathcal{C}\mathcal{C}$, thence $\operatorname{ind}(\widetilde{\mathcal{C}}\mathcal{C}) = \operatorname{ind}(\mathcal{C}\mathcal{C}) = 0$ and:

$$\operatorname{ind}(\mathcal{I}_{\mathcal{C}}\mathcal{C}) = \operatorname{ind}(\eth_{\mathcal{I}_{\mathcal{C}}}) = \operatorname{ind}(\eth_{\widetilde{\mathcal{C}}}) = \operatorname{ind}(\widetilde{\mathcal{C}}\mathcal{C}) = 0.$$

□

REMARK 2.4.13. By Lemma 2.4.12, we can always consider the Calderón to be diagonal with respect to the boundary decomposition. In other words, the index and its additivity depend only on the diagonal components of the operator.

2.4.1. An additive formula for manifolds with boundary. Let us go back to the Example 2.4.1 and consider two oriented manifolds X_i, $i = 1, 2$, such that $\partial X_i = Y_{i-1}^- \sqcup Y_i$ and at least one between Y_0 and Y_2 is non-empty. Let $\eth_i$ be the Dirac operators associated to the Clifford module bundles $E_i \to X_i$ such that they are consistent in a collar neighbourhood of Y_1, and let $\eth$ the Dirac operator associated to $X = X_1 \cup_{Y_1} X_2$ via gluing.

In order to define the Calderón projectors, we only need that each of the manifolds involved embeds smoothly into a closed manifold. We can therefore consider as a *common* closed manifold, the closed double $\widetilde{X}$ of X; thus, X_i embeds smoothly in X, for $i = 1, 2$, and so does X in $\widetilde{X}$ (Figure 1).

Let $\widetilde{\eth}$ be the invertible double of $\eth$ and $\widetilde{E}$ the double Clifford module bundle. Then $\widetilde{\eth}$ is an invertible extension to $\widetilde{X}$ of $\eth$, $\eth_1$, and $\eth_2$. Therefore, by Theorem 2.2.8, it is used to define the Poisson operators and Calderón projectors relative to X, X_1 and X_2, respectively. We consider restriction maps $r_i^\pm : H^s(\widetilde{X}, \widetilde{E}) \to H^s(X_i^\pm, E_i^\pm)$, with $i = 1, 2$ and $X_i^+ := X_i$, and trace maps:

$$\gamma_0^\pm : H^s(X_1^\mp, E_1^\mp) \to H^{s-\frac{1}{2}}(Y_0, E_0'), \qquad \gamma_2^\pm : H^s(X_2^\pm, E_2^\pm) \to H^{s-\frac{1}{2}}(Y_2, E_2'),$$
$$\gamma_1^+ : H^s(X_1^+, E_1^+) \to H^{s-\frac{1}{2}}(Y_1, E_1'), \qquad \gamma_1^- : H^s(X_2^+, E_2^+) \to H^{s-\frac{1}{2}}(Y_1, E_1').$$

As we have seen, we can write $\gamma_{\partial X_i^\pm} = \begin{pmatrix} \gamma_{i-1}^\mp \\ \gamma_i^\pm \end{pmatrix}$ with respect to the decomposition $H^s(\partial X_i, E_{|\partial X_i}) = H^s(Y_{i-1}, E_{i-1}') \oplus H^s(Y_i, E_i')$.

THEOREM 2.4.14. The following operator over $H^{s-\frac{1}{2}}(Y_1, E_1')$:

$$\gamma_1^+ r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 + \gamma_1^- r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 - I \tag{2.4.1}$$

is smoothing.

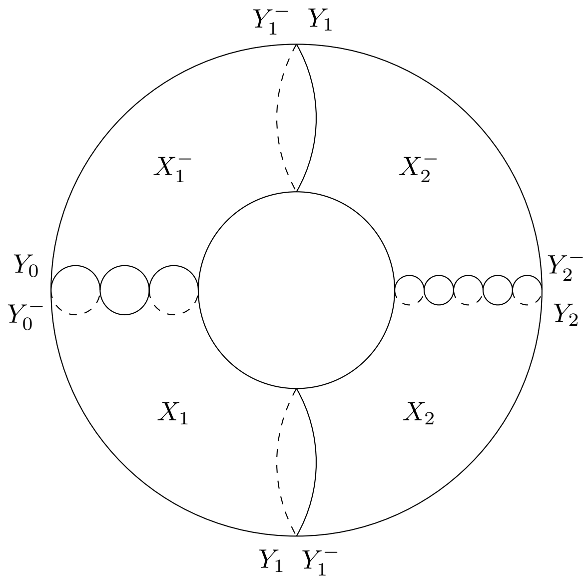

FIGURE 1. The double $\widetilde{X}$ of $X = X_1 \cup_{Y_1} X_2$.

PROOF. Let $\mathcal{C}_1^{\pm}$ be the Calderón projectors associated to X_1:

$$\mathcal{C}_1^{\pm} : H^{s-\frac{1}{2}}(Y_0, E_0') \oplus H^{s-\frac{1}{2}}(Y_1, E_1') \to H^{s-\frac{1}{2}}(Y_0, E_0') \oplus H^{s-\frac{1}{2}}(Y_1, E_1').$$

Set $r^+ := r_1^+$ and let $r^- : H^s(\widetilde{X}, \widetilde{E}) \to H^s(X_-, E_-)$ the restriction to $X_- := \overline{\widetilde{X} \setminus X_1}$ and $E_- := \widetilde{E}_{|X_-}$. Then we have:

$$\mathcal{C}_1^{\pm} = \begin{pmatrix} \gamma_0^{\mp} \\ \gamma_i^{\pm} \end{pmatrix} r^{\pm} \widetilde{\eth}^{-1} \left(\widetilde{\gamma}_0^* \sigma_0, \widetilde{\gamma}_1^* \sigma_1 \right) = \begin{pmatrix} \gamma_0^{\mp} r^{\pm} \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_0^{\mp} r^{\pm} \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \\ \gamma_1^{\pm} r^{\pm} \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_1^{\pm} r^{\pm} \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \end{pmatrix}.$$

Therefore, by the following equalities:

$$\gamma_0^- r^+ = \gamma_0^- r_1^+, \qquad \gamma_0^+ r^- = \gamma_0^+ r_1^-, \qquad \gamma_1^+ r^+ = \gamma_1^+ r_1^+, \qquad \gamma_1^- r^- = \gamma_1^- r_2^+,$$

we can write $\mathcal{C}_1^{\pm}$ in terms of $r_i^{\pm}$ as:

$$\mathcal{C}_1^+ = \begin{pmatrix} \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \\ \gamma_1^+ r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_1^+ r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{00}^+ & \mathcal{C}_{01}^+ \\ \mathcal{C}_{10}^+ & \mathcal{C}_{11}^+ \end{pmatrix} \text{ and}$$

$$\mathcal{C}_1^- = \begin{pmatrix} \gamma_0^+ r_1^- \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_0^+ r_1^- \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \\ \gamma_1^- r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_1^- r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{00}^- & \mathcal{C}_{01}^- \\ \mathcal{C}_{10}^- & \mathcal{C}_{11}^- \end{pmatrix}.$$

Let $\varphi_j \in H^{s-\frac{1}{2}}(Y_j, E_j')$, $j = 0, 1$. Then, since $\mathcal{C}_1^+ + \mathcal{C}_1^- = I$, we obtain:

$$(\mathcal{C}_{00}^+ + \mathcal{C}_{00}^- - I)\varphi_0 + (\mathcal{C}_{01}^+ + \mathcal{C}_{01}^-)\varphi_1 = 0$$

$$(\mathcal{C}_{10}^+ + \mathcal{C}_{10}^-)\varphi_0 + (\mathcal{C}_{11}^+ + \mathcal{C}_{11}^- - I)\varphi_1 = 0.$$

Now, $\mathcal{C}^{\pm}_{jk}$ are smoothing operators when $j \neq k$; thus we conclude that $\mathcal{C}^{+}_{jj} + \mathcal{C}^{-}_{jj} - I$ are smoothing operators.

□

Let $\mathcal{C}_1 := \mathcal{C}_1^+$, $\mathcal{C}_2 := \mathcal{C}_2^+$ and $\mathcal{C} := \mathcal{C}^+$ be Calderón projectors for $\eth_1$, $\eth_2$ and $\eth$ respectively:

$$\begin{aligned}
\mathcal{C}_1 :& H^{s-\frac{1}{2}}(Y_0, E_{|Y_0}) \oplus H^{s-\frac{1}{2}}(Y_1, E_{|Y_1}) \to H^{s-\frac{1}{2}}(Y_0, E_{|Y_0}) \oplus H^{s-\frac{1}{2}}(Y_1, E_{|Y_1}) \\
\mathcal{C}_2 :& H^{s-\frac{1}{2}}(Y_1, E_{|Y_1}) \oplus H^{s-\frac{1}{2}}(Y_2, E_{|Y_2}) \to H^{s-\frac{1}{2}}(Y_1, E_{|Y_1}) \oplus H^{s-\frac{1}{2}}(Y_2, E_{|Y_2}) \\
\mathcal{C} :& H^{s-\frac{1}{2}}(Y_0, E_{|Y_0}) \oplus H^{s-\frac{1}{2}}(Y_2, E_{|Y_2}) \to H^{s-\frac{1}{2}}(Y_0, E_{|Y_0}) \oplus H^{s-\frac{1}{2}}(Y_2, E_{|Y_2}).
\end{aligned}$$

Therefore, they are defined as:

$$\begin{aligned}
\mathcal{C}_1 &= \begin{pmatrix} \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \\ \gamma_1^+ r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_1^+ r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{0,0} & \mathcal{C}_{0,1} \\ \mathcal{C}_{1,0} & \mathcal{C}_{1,1} \end{pmatrix} \\
\mathcal{C}_2 &= \begin{pmatrix} \gamma_1^- r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 & \gamma_1^- r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_2^* \sigma_2 \\ \gamma_2^+ r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_1^* \sigma_1 & \gamma_2^+ r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_2^* \sigma_2 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{1,1} & \mathcal{D}_{1,2} \\ \mathcal{D}_{2,1} & \mathcal{D}_{2,2} \end{pmatrix} \\
\mathcal{C} &= \begin{pmatrix} \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_0^- r_1^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_2^* \sigma_2 \\ \gamma_2^+ r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_0^* \sigma_0 & \gamma_2^+ r_2^+ \widetilde{\eth}^{-1} \widetilde{\gamma}_2^* \sigma_2 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{0,0} & \mathcal{E}_{0,2} \\ \mathcal{E}_{2,0} & \mathcal{D}_{2,2} \end{pmatrix},
\end{aligned}$$

where $\mathcal{C}_{i,j} := \mathcal{C}^+_{i,j}$, $i, j \in \{0, 1\}$ as in the proof of Theorem 2.4.14, and $\mathcal{D}_{1,1} = \mathcal{C}^-_{1,1}$ by inspection.

THEOREM 2.4.15. Let $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}$ be well-posed boundary conditions respectively for $\eth_1$, $\eth_2$, and $\eth$, such that:

$$\mathcal{P}_1 = \begin{pmatrix} \mathcal{P}_{0,0} & \mathcal{P}_{0,1} \\ \mathcal{P}_{1,0} & \mathcal{P}_{1,1} \end{pmatrix}, \mathcal{P}_2 = \begin{pmatrix} \widetilde{\mathcal{P}}_{1,1} & \mathcal{P}_{1,2} \\ \mathcal{P}_{2,1} & \mathcal{P}_{2,2} \end{pmatrix}, \text{ and } \mathcal{P} = \begin{pmatrix} \mathcal{P}_{0,0} & \mathcal{P}_{0,2} \\ \mathcal{P}_{2,0} & \mathcal{P}_{2,2} \end{pmatrix}.$$

Then

$$\operatorname{ind}(\mathcal{P}\mathcal{C}) = \operatorname{ind}(\mathcal{P}_1\mathcal{C}_1) + \operatorname{ind}(\mathcal{P}_2\mathcal{C}_2) + \operatorname{ind}\left(\mathcal{P}_{1,1}\widetilde{\mathcal{P}}^{\perp}_{1,1}\right).$$

PROOF. $\mathcal{P}_1\mathcal{C}_1$ is Fredholm and a smooth perturbation of a diagonal operator:

$$\begin{aligned}
\mathcal{P}_1\mathcal{C}_1 &= \begin{pmatrix} \mathcal{P}_{0,0}\mathcal{C}_{0,0} + \mathcal{P}_{0,1}\mathcal{C}_{1,0} & \mathcal{P}_{0,0}\mathcal{C}_{0,1} + \mathcal{P}_{0,1}\mathcal{C}_{1,1} \\ \mathcal{P}_{1,0}\mathcal{C}_{0,0} + \mathcal{P}_{1,1}\mathcal{C}_{1,1} & \mathcal{P}_{1,0}\mathcal{C}_{0,1} + \mathcal{P}_{1,1}\mathcal{C}_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{P}_{0,0}\mathcal{C}_{0,0} & 0 \\ 0 & \mathcal{P}_{1,1}\mathcal{C}_{1,1} \end{pmatrix} + \begin{pmatrix} \mathcal{P}_{0,1}\mathcal{C}_{1,0} & \mathcal{P}_{0,0}\mathcal{C}_{0,1} + \mathcal{P}_{0,1}\mathcal{C}_{1,1} \\ \mathcal{P}_{1,0}\mathcal{C}_{0,0} + \mathcal{P}_{1,1}\mathcal{C}_{1,1} & \mathcal{P}_{1,0}\mathcal{C}_{0,1} \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\operatorname{ind}(\mathcal{P}_1\mathcal{C}_1) = \operatorname{ind}\begin{pmatrix} \mathcal{P}_{0,0}\mathcal{C}_{0,0} & 0 \\ 0 & \mathcal{P}_{1,1}\mathcal{C}_{1,1} \end{pmatrix} = \operatorname{ind}(\mathcal{P}_{0,0}\mathcal{C}_{0,0}) + \operatorname{ind}(\mathcal{P}_{1,1}\mathcal{C}_{1,1}).$$

Analogously,

$$\operatorname{ind}(\mathcal{P}_2\mathcal{C}_2) = \operatorname{ind}(\widetilde{\mathcal{P}}_{1,1}\mathcal{D}_{1,1}) + \operatorname{ind}(\mathcal{P}_{2,2}\mathcal{D}_{2,2})$$

$$\operatorname{ind}(\mathcal{P}\mathcal{C}) = \operatorname{ind}(\mathcal{P}_{0,0}\mathcal{C}_{0,0}) + \operatorname{ind}(\mathcal{P}_{2,2}\mathcal{D}_{2,2}),$$

thus

$$\operatorname{ind}(\mathcal{P}_1\mathcal{C}_1) + \operatorname{ind}(\mathcal{P}_2\mathcal{C}_2) - \operatorname{ind}(\mathcal{P}\mathcal{C}) = \operatorname{ind}(\mathcal{P}_{1,1}\mathcal{C}_{1,1}) + \operatorname{ind}(\widetilde{\mathcal{P}}_{1,1}\mathcal{D}_{1,1}). \tag{2.4.2}$$

Now, by iv) of Proposition 2.2.6, $\operatorname{ind}(\widetilde{\mathcal{P}}_{1,1}\mathcal{D}_{1,1}) = \operatorname{ind}(\mathcal{D}_{1,1}^{\perp}\widetilde{\mathcal{P}}_{1,1}^{\perp})$, and since $\mathcal{C}_{1,1} + \mathcal{D}_{1,1} - I$ is smoothing by Theorem 2.4.14, we obtain

$$\operatorname{ind}(\mathcal{D}_{1,1}^{\perp}\widetilde{\mathcal{P}}_{1,1}^{\perp}) = \operatorname{ind}(\mathcal{C}_{1,1}\widetilde{\mathcal{P}}_{1,1}^{\perp}).$$

Thus (2.4.2) becomes:

$$\operatorname{ind}(\mathcal{P}_1\mathcal{C}_1) + \operatorname{ind}(\mathcal{P}_2\mathcal{C}_2) - \operatorname{ind}(\mathcal{P}\mathcal{C}) = \operatorname{ind}(\mathcal{P}_{1,1}\mathcal{C}_{1,1}) + \operatorname{ind}(\mathcal{C}_{1,1}\widetilde{\mathcal{P}}_{1,1}^{\perp}) = \operatorname{ind}(\mathcal{P}_{1,1}\widetilde{\mathcal{P}}_{1,1}^{\perp}).$$

□

REMARK 2.4.16. For example, if we consider de Rham operators $\eth := (d+\delta)^+$ with relative boundary conditions on the boundaries, we have:

$$\operatorname{ind}(\mathcal{R}\mathcal{C}) = \operatorname{ind}(\mathcal{R}\mathcal{C}_1) + \operatorname{ind}(\mathcal{R}\mathcal{C}_2) + \operatorname{ind}(\mathcal{R}_{Y_1}^{\perp}\mathcal{R}_{Y_1}).$$

Since $\operatorname{ind}(\mathcal{R}_{Y_1}^{\perp}\mathcal{R}_{Y_1}) = \chi(Y_1)$, the above formula reduces to

$$\operatorname{ind}(\mathcal{R}\mathcal{C}) = \operatorname{ind}(\mathcal{R}\mathcal{C}_1) + \operatorname{ind}(\mathcal{R}\mathcal{C}_2). \tag{2.4.3}$$

if n is even.

2.4.2. Index and trace class operators. When the boundary conditions are trace class operators, the index can be interpreted in terms of the trace.

LEMMA 2.4.17 (Lemma 3.8, [**72**]). Let $H = H_+ \oplus H_-$ be a polarized Hilbert space, with $H_\pm$ infinite-dimensional, and let $\Pi_\pm$ denote the orthogonal projections onto $H_\pm$. Let P_0, P_1 be projections on H such that $P_i - \Pi_+$ is trace-class on H, $i = 0, 1$. Then $P_0 - P_1$ is trace-class on H and $P_1P_0 : \operatorname{ran}(P_0) \to \operatorname{ran}(P_1)$ is a Fredholm operator, and the index satisfies:

$$\operatorname{ind}(P_1P_0) = \operatorname{Tr}_H(P_0 - P_1). \tag{2.4.4}$$

REMARK 2.4.18. Equality (2.4.4) applies to projections $\mathcal{P}_0, \mathcal{P}_1 \in \Psi^0(Y, E')$ such that $\mathcal{P}_0 - \mathcal{P}_1 \in \Psi^{-\infty}(Y, E')$, with $\mathrm{Tr}_H = \mathrm{Tr}_{\Psi^{-\infty}(Y,E')} =: \mathrm{Tr}$ the classical trace of smoothing pseudodifferential operators:

$$\mathrm{ind}(\mathcal{P}_1\mathcal{P}_0) = \mathrm{Tr}(\mathcal{P}_0 - \mathcal{P}_1). \tag{2.4.5}$$

In particular, if $Q \in \ker(\mathrm{Tr}) = [\Psi^{-\infty}(Y, E'), \Psi^{-\infty}(Y, E')]$,

$$\mathrm{ind}(\mathcal{P}_1\mathcal{P}_0) = \mathrm{Tr}(\mathcal{P}_0 - \mathcal{P}_1 + \mathcal{Q}) = \mathrm{ind}(\mathcal{P}_1(\mathcal{P}_0 + \mathcal{Q})) = \mathrm{ind}((\mathcal{P}_1 - \mathcal{Q})\mathcal{P}_0),$$

i.e. index is stable with respect to commutators of smoothing pseudodifferential operators.

REMARK 2.4.19 (From Remark 18.17, [**10**]). Let $\mathcal{P} \in \Psi^0(Y, E')$ be a well-posed boundary condition. Then $\mathcal{C}^+ - \mathcal{I}_{\mathcal{C}^+\mathcal{P}^*}$ and $\mathcal{I}_{\mathcal{P}} - \mathcal{I}_{\mathcal{P}\mathcal{C}^+}$ are smoothing pseudodifferential operators.

In fact, since $\mathrm{ran}(\mathcal{I}_{\mathcal{C}^+\mathcal{P}^*}) = \mathrm{ran}(\mathcal{C}^+\mathcal{P}^*) = (\ker((\mathcal{C}^+\mathcal{P}^*)^*))^\perp = (\ker(P\mathcal{C}^+))^\perp$, we have the orthogonal decomposition $\mathrm{ran}(\mathcal{C}^+) = \mathrm{ran}(\mathcal{C}^+\mathcal{P}^*) \oplus \ker(\mathcal{P}\mathcal{C}^+)$. Therefore $\mathcal{C}^+ - \mathcal{I}_{\mathcal{C}^+\mathcal{P}}$ is the orthogonal projection onto $\ker(\mathcal{P}\mathcal{C}^+)$, which is finite dimensional since $\mathcal{P}\mathcal{C}^+$ is Fredholm. Hence $\mathcal{C}^+ - \mathcal{I}_{\mathcal{C}^+\mathcal{P}^*}$ is a finite rank operator, and as such it is smoothing. Analogously for $\mathcal{I}_{\mathcal{P}} - \mathcal{I}_{\mathcal{P}\mathcal{C}^+}$, since $\mathrm{ran}(\mathcal{I}_{\mathcal{P}}) = \mathrm{ran}(\mathcal{P}\mathcal{C}^+) \oplus \ker(\mathcal{C}^+\mathcal{P}^*)$.

THEOREM 2.4.20. Let $\mathcal{P} \in \Psi^0(Y, E')$ be a well-posed boundary condition and $\varphi : L^2(Y, E') \to L^2(Y, E')$ an isomorphism extending $\mathrm{ran}(\mathcal{C}^+\mathcal{Q}^*) \cong \mathrm{ran}(\mathcal{Q}\mathcal{C}^+)$. Then:

$$\mathrm{ind}(\mathcal{P}\mathcal{C}^+) = \mathrm{Tr}(\mathcal{C}^+ - \widetilde{\mathcal{I}_{\mathcal{P}}}), \qquad \widetilde{\mathcal{I}_{\mathcal{P}}} := \varphi^{-1}\mathcal{I}_{\mathcal{P}}\varphi. \tag{2.4.6}$$

PROOF. Set $\mathcal{C} := \mathcal{C}^+$. From (2.2.2) and (2.4.5), we have:

$$\mathrm{ind}(\mathcal{P}\mathcal{C}) = \mathrm{Tr}(\mathcal{C} - \mathcal{I}_{\mathcal{C}\mathcal{P}^*}) - \mathrm{Tr}(\mathcal{I}_{\mathcal{P}} - \mathcal{I}_{\mathcal{P}\mathcal{C}})$$

Now, recall that $\mathcal{Q}_{|\mathrm{ran}(\mathcal{C}\mathcal{Q}^*)} : \mathrm{ran}(\mathcal{C}\mathcal{Q}^*) \overset{\cong}{\to} \mathrm{ran}(\mathcal{Q}\mathcal{C})$ for a general well-posed boundary condition $\mathcal{Q}$ (Proposition 18.16, [**10**]). Let φ be an isomorphism extending $\mathrm{ran}(\mathcal{C}\mathcal{Q}^*) \cong \mathrm{ran}(\mathcal{Q}\mathcal{C})$. Then, $\mathcal{I}_{\mathcal{C}\mathcal{P}^*} = \varphi^{-1}\mathcal{I}_{\mathcal{P}\mathcal{C}}\varphi$ and by the invariance of the trace:

$$\begin{aligned}
\mathrm{ind}(\mathcal{P}\mathcal{C}) &= \mathrm{Tr}(\mathcal{C} - \mathcal{I}_{\mathcal{C}\mathcal{P}^*}) - \mathrm{Tr}(\mathcal{I}_{\mathcal{P}} - \mathcal{I}_{\mathcal{P}\mathcal{C}}) \\
&= \mathrm{Tr}(\mathcal{C} - \mathcal{I}_{\mathcal{C}\mathcal{P}^*}) - \mathrm{Tr}(\varphi^{-1}\mathcal{I}_{\mathcal{P}}\varphi - \varphi^{-1}\mathcal{I}_{\mathcal{P}\mathcal{C}}\varphi) \\
&= \mathrm{Tr}(\mathcal{C} - \mathcal{I}_{\mathcal{C}\mathcal{P}^*} - \varphi^{-1}\mathcal{I}_{\mathcal{P}}\varphi + \varphi^{-1}\mathcal{I}_{\mathcal{P}\mathcal{C}}\varphi) = \mathrm{Tr}(\mathcal{C} - \varphi^{-1}\mathcal{I}_{\mathcal{P}}\varphi).
\end{aligned}$$

□

2.5. LogTQFT formulation of the Euler Characteristic

As for the topological signature in [**72**], we can define a log-functor on the category of even dimensional bordisms $\mathbf{Cob}_{2n}$ whose log-determinant will be the relative Euler characteristic of the cobordism.

Let X be a $2n$-dimensional oriented manifold with boundary Y and let us consider the de Rham operator $\eth := d + \delta : \Omega(X) \to \Omega(X)$ of Example 2.1.2, i.e. $E = \Lambda(X)$ and $E' = \Lambda(X)_{|Y}$.

Let us set $L^2\Omega(X) := L^2(X, \Lambda(X))$ and $H^s\Omega(X) := H^s(X, \Lambda(X))$, and consider *relative* and *absolute* boundary conditions for $\eth$, i.e. the orthogonal projections $\mathcal{R}, \mathcal{A} \in \Psi^0\Lambda(X)_{|Y} := \Psi^0(Y, \Lambda(X)_{|Y})$ of Definition 0.3.2:

$$\begin{aligned} \mathcal{R} : \Omega(X)_{|Y} &\to \Omega(Y) & \qquad \mathcal{A} : \Omega(X)_{|Y} &\to \Omega(Y) \\ \omega_{|Y} &\mapsto \omega_1 & \omega_{|Y} &\mapsto \omega_2. \end{aligned}$$

Consider $s_1, s_2 \in \Omega(X)$ such that $\mathcal{R}\gamma s_i = 0$ or $\mathcal{A}\gamma s_i = 0$. Then, by Green's formula, $\mathcal{R}$ and $\mathcal{A}$ are self-adjoint boundary conditions for $\eth$, i.e.:

$$\langle \eth s_1, s_2 \rangle - \langle s_1, \eth s_2 \rangle = -\langle \sigma\gamma s_1, \gamma s_2 \rangle = 0.$$

Therefore, if we want a non-vanishing index, we need to consider a $\mathbb{Z}_2$-grading of $\Omega(X)$. Let the grading be the one of Example 2.2.1, and consider the associated Dirac operator $\eth^+ := \eth_{|\Omega^+(X)}$, i.e.

$$\eth^+ = (d + \delta)^+ : \Omega^+(X) \to \Omega^-(X) \qquad \Omega^\pm := C^\infty(X, \Lambda^\pm(X)).$$

We remark that $\eth^+$ is not self-adjoint, but $(\eth^+)^* = \eth^-$.

Proposition 2.5.1. Relative and absolute boundary conditions $\mathcal{R}, \mathcal{A}$ are well-posed boundary condition for the de Rham operator $\eth^+$.

Proof. $\mathcal{R}$ and $\mathcal{A}$ are truly orthogonal projection at the bundle level, thus independent of (y, ζ), and their ranges are closed for each $s \in \mathbb{R}$, since one projection is the complement of the other. Moreover, Lemma 4.1.1, [**23**], shows that $\mathcal{R} : N_\pm(y, \zeta) \to \mathrm{ran}(\mathcal{R}) = \Lambda^+(Y)$ and $\mathcal{A} : N_\pm(y, \zeta) \to \mathrm{ran}(\mathcal{A}) = \Lambda^-(Y)$ are isomorphisms. $\square$

Remark 2.5.2. In particular, $\mathcal{R}$ and $\mathcal{A}$ are *local* well-posed boundary conditions (Example 3.19, [?baBa]). This places the complex $d + \delta : \Omega(X) \to \Omega(X)$ in a rather special situation, since there are no local well-posed conditions for the other classical elliptic complexes: the signature, the spin and the Dolbeaux complex (Lemma 4.1.6, [**23**]).

The realization of $\eth^+$ with respect to relative boundary conditions,

$$\eth^+_{\mathcal{R}} : \mathrm{dom}\left(\eth^+_{\mathcal{R}}\right) \to L^2\Omega^-(X)$$
$$\mathrm{dom}\left(\eth^+_{\mathcal{R}}\right) = \{\omega \in H^1\Omega^+(X) \mid \mathcal{R}\gamma\omega = 0\}$$

which is a Fredholm operator by Theorem 2.2.3.

LEMMA 2.5.3. For $\ker \eth^k_{\mathcal{R}} = \{\omega \in H^1\Omega^k(X) | \; \eth\omega = 0, \; \mathcal{R}\gamma\omega = 0\}$, we have:

i) $\ker \eth^+_{\mathcal{R}} = \bigoplus_{k=0}^n \ker \eth^{2k}_{\mathcal{R}}$ (Lemma 4.1.2, [**23**]);

ii) for $H^k_{\mathcal{R}}(X)$ the relative de Rham cohomology of §0.4, (Corollary 2.6.2, [**68**]):

$$\ker \eth^k_{\mathcal{R}} \cong H^k_{\mathcal{R}}(X). \tag{2.5.1}$$

PROPOSITION 2.5.4. Let $\mathcal{C} := \mathcal{C}^+$ denote the Calderón projector for $\eth^+$. Then:

$$\mathrm{Tr}(\mathcal{C} - \widetilde{\mathcal{R}}) = \chi(X, Y).$$

PROOF. We only have to combine all the previous results together. By §0.4, the relative Euler characteristic can be defined as $\chi(X,Y) = \sum_k (-1)^k \dim H^k_{\mathcal{R}}(X)$. Hence,

$$\chi(X,Y) \overset{(2.5.1)}{=} \sum_{k=0}^{2n} (-1)^k \dim \ker \eth^k_{\mathcal{R}} = \mathrm{ind}\eth^+_{\mathcal{R}} \overset{(2.2.3)}{=} \mathrm{ind}(R\mathcal{C}) \overset{(2.4.6)}{=} \mathrm{Tr}(\mathcal{C} - \widetilde{\mathcal{R}}).$$

□

We finally have all the ingredient to define a LogTQFT associated to the relative Euler characteristic. Let us consider the strict functor $F_{-\infty} : \mathbf{Cob}^*_{2n} \to \mathbb{C}\text{-}\mathbf{Alg}$ of Example 1.4.38, i.e.

$$F_{-\infty}(Y) := \Psi^{-\infty}(Y, \Lambda(X)_{|Y}) \qquad Y \in \mathrm{obj}(\mathbf{Cob}^*_{2n}) \text{ such that } Y = \partial X.$$

By Lemma 1.4.39, $(F_{-\infty}, \mathrm{Tr})$, with Tr the classical trace, is an unoriented tracial monoidal representation.

Let W be a representative of a morphism $\overline{W} \in \mathrm{mor}_{\mathbf{Cob}_{2m}}(M_0, M_1)$, where M_0, M_1 are not both empty. As we have seen in the Example 1.4.5, it comes with an orientation-preserving diffeomorphism $\kappa : \partial W \to M_0^- \sqcup M_1$ that induces the isomorphism $\kappa_\sharp : F_{-\infty}(\partial W) \to F_{-\infty}(M_0 \sqcup M_1)$, by Lemma 1.4.39. Consider the following simplicial map:

$$\begin{gathered}\log^\chi : \mathcal{N}\mathbf{Cob}_{2n} \longrightarrow F_{-\infty,\Pi}(\mathbf{Cob}^*_{2n}) \\ \log^\chi_{M_0\sqcup M_1}(\overline{W}) := \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp(\mathcal{C}_W - \widetilde{\mathcal{R}}_{\partial W}) \in F_{-\infty,\Pi}(M_0 \sqcup M_1)\end{gathered} \tag{2.5.2}$$

THEOREM 2.5.5. (2.5.2) defines a LogTQFT. In other words, for $X = X_1 \cup_f X_2$ with $\partial X_i \cong M_{i-1}^- \sqcup M_i$ and f a diffeomorphism used for the gluing, (1.4.4) holds, i.e. in $F_{-\infty,\Pi}(M_0 \sqcup M_1 \sqcup M_2)$

$$\widetilde{\eta}_{M_1} \log^{\chi}_{M_0 \sqcup M_2}(\overline{X}) = \widetilde{\eta}_{M_2} \log^{\chi}_{M_0 \sqcup M_1}(\overline{X_1}) + \widetilde{\eta}_{M_0} \log^{\chi}_{M_1 \sqcup M_2}(\overline{X_2}) \tag{2.5.3}$$

PROOF. First of all, assume $X = X_1 \cup_{Y_1} X_2$, i.e. f is the identity and X_1 and X_2 have a common boundary. Then, by (2.5.6) and (1.4.7), we have that:

$$\operatorname{ind}(\mathcal{R}_{\partial X}\mathcal{C}_X) = \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_2}\left(\log^{\chi}_{M_0 \sqcup M_2}(\overline{X})\right) = \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{M_1} \log^{\chi}_{M_0 \sqcup M_2}(\overline{X})\right)$$

$$\operatorname{ind}(\mathcal{R}_{\partial X_1}\mathcal{C}_{X_1}) = \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{M_2} \log^{\chi}_{M_0 \sqcup M_1}(\overline{X}_1)\right) \text{ and}$$

$$\operatorname{ind}(\mathcal{R}_{\partial X_2}\mathcal{C}_{X_2}) = \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{M_0} \log^{\chi}_{M_1 \sqcup M_2}(\overline{X}_2)\right).$$

Therefore, by (2.4.3) and linearity of the trace:

$$\widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{M_1} \log^{\chi}_{M_0 \sqcup M_2}(\overline{X}) - \widetilde{\eta}_{M_2} \log^{\chi}_{M_0 \sqcup M_1}(\overline{X}_1) - \widetilde{\eta}_{M_0} \log^{\chi}_{M_1 \sqcup M_2}(\overline{X}_2)\right) = 0.$$

Since $\widetilde{\operatorname{Tr}}$ is an isomorphism onto $\mathbb{C}$, we obtain, in $F_{-\infty,\Pi}(M_0 \sqcup M_1 \sqcup M_2)$,

$$\widetilde{\eta}_{M_1} \log^{\chi}_{M_0 \sqcup M_2}(\overline{X}) - \widetilde{\eta}_{M_2} \log^{\chi}_{M_0 \sqcup M_1}(\overline{X}_1) - \widetilde{\eta}_{M_0} \log^{\chi}_{M_1 \sqcup M_2}(\overline{X}_2) = 0.$$

Let now assume $\partial X_1 = Y_0^- \sqcup Y_1$, $\partial X_1 = \widetilde{Y}_1^- \sqcup Y_2$, and $f : Y_1 \to \widetilde{Y}_1$ a diffeomorphism. Let $X_f := X_1 \cup_f X_2$ be the resulting glued manifold. In a collar neighborhood of Y_1 and $\widetilde{Y}_1$ the respective Dirac operators are compatible by local invariance of smooth forms with respect to diffeomorphisms. Hence, we can define a Dirac operator on X_f and by Theorem 25.4 of [**10**] the additive formula for the index is the same.

□

PROPOSITION 2.5.6. The logarithm defined in (2.5.2) depends only on the oriented bordism class $\overline{W}$ and has log-determinant

$$\widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W})\right) = \chi(W, \partial W).$$

PROOF. Form Proposition 2.5.4, $\chi(W, \partial W) = \operatorname{Tr}_{\partial W}\left(\mathcal{C}_W - \widetilde{\mathcal{R}}_{\partial W}\right)$. Thence:

$$\begin{aligned}
\operatorname{Tr}_{\partial W}\left(\mathcal{C}_W - \mathcal{R}_{\partial W}\right) &\overset{(1.4.8)}{=} \operatorname{Tr}_{M_0 \sqcup M_1}(\kappa_\sharp(\mathcal{C}_W - \mathcal{R}_{\partial W})) \\
&\overset{(1.4.7)}{=} \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\pi_{M_0 \sqcup M_1}\left(\kappa_\sharp(\mathcal{C}_W - \mathcal{R}_{\partial W})\right)\right) \\
&\overset{(2.5.2)}{=} \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W})\right).
\end{aligned}$$

Let $W, W' \in \overline{W}$, thus $\chi(W, \partial W) = \chi(W', \partial W')$ and

$$\widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W}) - \log^{\chi}_{M_0 \sqcup M_1}(\overline{W'})\right) = 0,$$

i.e. $\log^{\chi}_{M_0 \sqcup M_1}(\overline{W}) - \log^{\chi}_{M_0 \sqcup M_1}(\overline{W'}) = 0$ in $F_{-\infty,\Pi}(M_0 \sqcup M_1)$.

□

COROLLARY 2.5.7. The relative Euler characteristic is additive with respect to composition of cobordisms, i.e. identity (0.4.4)

$$\chi(X, \partial X) = \chi(X_1, \partial X_1) + \chi(X_2, \partial X_2).$$

PROPOSITION 2.5.8. (2.5.2) is unoriented.

PROOF. Since $\chi(W, \partial W) = \chi(W^-, \partial W^-)$ and $F_{-\infty}$ is unoriented, then:

$$\widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W})\right) = \widetilde{\operatorname{Tr}}_{M_0 \sqcup M_1}\left(\log^{\chi}_{M_0 \sqcup M_1}(\overline{W'})\right)$$

Hence, we conclude as in the proof of Proposition 2.5.6.

□

REMARK 2.5.9. By Corollary 1.4.42, (2.5.2) on $\mathbf{Cob}_2$ needs only to be defined on $\overline{D}$. In fact, as $\widetilde{\operatorname{Tr}}\left(\log^{\chi}_{S^1} \overline{D}\right) = \chi(D, S^1) = \chi(D) = 1$, for all other compact surfaces we obtain:

$$\widetilde{\operatorname{Tr}}\left(\log^{\chi}_{S^1} \overline{\Sigma}_0\right) = \chi(\Sigma_0) \cdot \widetilde{\operatorname{Tr}}\left(\log^{\chi}_{S^1} \overline{D}\right) = \chi(\Sigma_0),$$

$$\widetilde{\operatorname{Tr}}\left(\log^{\chi}_{S^1 \sqcup S^1 \sqcup S^1} \overline{\Sigma}_g\right) = \chi(\Sigma_g) \quad \text{and} \quad \widetilde{\operatorname{Tr}}\left(\log_{\bigsqcup_k S^1} \overline{\Sigma}_{g,k}\right) = \chi(\Sigma_g) - k = \chi(\Sigma_{g,k}),$$

i.e. the results are consistent.

Let us go back to the setting of Theorem 2.5.5. If $Y_0 = Y_2 = \emptyset$, we have that $X = X_1 \cup_{Y_1} X_2$ is closed and we do not have a boundary to associate boundary conditions to. Then we extend the definition of (2.5.2) to this case by setting:

$$\log^{\chi}_{M_1}(\overline{X}) := \pi_{M_1} \circ \kappa_{\sharp}\left(\mathcal{C}_1 - \mathcal{C}_2^{\perp}\right)\right) \in F_{-\infty,\Pi}(M_1), \quad \text{where} \quad \mathcal{C}_i := \mathcal{C}^{+}_{X_i}.$$

PROPOSITION 2.5.10. $\widetilde{\operatorname{Tr}}_{M_1}\left(\log^{\chi}_{M_1}(\overline{X})\right) = \chi(X)$.

PROOF.

$$\begin{aligned}\widetilde{\operatorname{Tr}}_{M_1}\left(\log^{\chi}_{Y_1}(\overline{X})\right) &= \widetilde{\operatorname{Tr}}_{M_1}\left(\pi_{M_1} \circ \kappa_{\sharp}\left(\mathcal{C}_1 - \mathcal{C}_2^{\perp}\right)\right) = \operatorname{Tr}_{M_1}\left(\kappa_{\sharp}\left(\mathcal{C}_1 - \mathcal{C}_2^{\perp}\right)\right) \\ &= \operatorname{Tr}_{Y_1}\left(\mathcal{C}_1 - \mathcal{C}_2^{\perp}\right) = \operatorname{ind}\left(\mathcal{C}_2^{\perp}\mathcal{C}_1\right) \overset{(2.3.1)}{=} \operatorname{ind}(\eth^{+}_{X}) = \chi(X).\end{aligned}$$

□

REMARK 2.5.11. Since the dimension is even, then $\chi(X, Y) = \chi(X)$ and we could have defined an equivalent LogTQFT through absolute boundary conditions. Also, as $\mathcal{R}$ is independent of the metric g^X (§0.4), $\chi(X, Y)$ is independent of the metric.

Part 2

Higher LogTQFTs

CHAPTER 3

Higher log-functors and cyclic homology

Let R be an associative ring. From Remark 1.4.17, the projection $R \to \frac{R}{[R,R]}$ defines a functor $\Pi : \mathbf{Ring} \to \frac{\mathbf{Ring}}{[\mathbf{Ring},\mathbf{Ring}]} \subset \mathbf{Ab}$. In fact, Π corresponds to the functor from the category of rings into the category of abelian groups that associates a ring to its first cyclic homology group.

Here we briefly present cyclic homology and cohomology in order to extend the concepts of tracial monoidal product representations, log-functors and logTQFTs.

3.1. Cyclic (co)homology and higher log-functors

Let R be a commutative ring and $\mathcal{A}$ be an (associative) R-algebra. We can define an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on the $(n+1)$-fold tensor product

$$\mathcal{A}^{\otimes n+1} := \mathcal{A} \otimes \cdots \otimes \mathcal{A}.$$

in the following way. If $t_n : \mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n+1}$ is the generator of $\mathbb{Z}/(n+1)\mathbb{Z}$, then on the generators of $\mathcal{A}^{\otimes n+1}$

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}).$$

Definition 3.1.1 (Definition 2.1.4, [**46**]). The *Hochschild boundary map* is the R-linear map $b_n : \mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n}$ such that:

$$\begin{aligned} b_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := &\sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

If $C_n^\lambda(\mathcal{A}) := \frac{\mathcal{A}^{\otimes n+1}}{\mathrm{im}(1-t_n)}$, then b_n restricts to $C_n^\lambda(\mathcal{A})$ (Lemma 2.1.1, [**46**]) and $(C_*^\lambda(\mathcal{A}), b)$ is the so-called *Connes' complex.*

Definition 3.1.2 (§2.1, [**46**]). *Cyclic homology* is the homology of Connes' complex[1]. We denote the n^{th} cyclic homology group by $HC_n(\mathcal{A})$, and we set $HC_*(\mathcal{A}) := \bigoplus_{n\geq 0} HC_n(\mathcal{A})$.

[1]There are several but equivalent definitions of cyclic homology. See Theorem 2.1.5, [**46**].

REMARK 3.1.3 (§2.1, [**46**]). If $f : \mathcal{A} \to \mathcal{B}$ is a morphism of R-algebras, then $f_* : HC_n(f) : HC_*(\mathcal{A}) \to HC_*(\mathcal{B})$ is a morphism of R-modules. Therefore HC_n is a functor from R-**Alg**, the category of R-algebras, to R-**Mod**.

EXAMPLE 3.1.4 (§2.1.12, [**46**]). For any ring R, $\frac{R}{[R,R]} \cong HC_0(R)$. In particular, if R is commutative, then $R \cong HC_0(R)$.

DEFINITION 3.1.5 (2.2.13, [**46**]). Let $\overline{C}_n^\lambda(\mathcal{A})$ be the quotient of $C_n^\lambda(\mathcal{A})$ by the sub-module generated by those $a_0 \otimes \cdots \otimes a_n$ such that $a_i = 1$ for some $i \in \{0, 1, \ldots, n\}$. Then $(\overline{C}_*^\lambda(\mathcal{A}), b)$ is a well-defined complex called *reduced Connes' complex*; its homology is called *reduced cyclic homology* and is denoted by $\overline{HC}_*(\mathcal{A})$.

DEFINITION 3.1.6. A pretracial monoidal product representation of a symmetric monoidal category $\mathbf{C}$, $F : \mathbf{C}^* \to \mathbf{Ring_{Add}}$, is said to be *higher pretracial* if in particular it is a monoidal product representation $F : \mathbf{C}^* \to R\text{-}\mathbf{Alg}$, i.e. $\forall c \in \mathrm{obj}(\mathbf{C})$, $F(c)$ are R-algebras, $\eta_{\otimes y} c$ are R-algebra homomorphisms and $\mu_\sigma c$ are R-algebra isomorphisms.

EXAMPLE 3.1.7. The category of R-modules R-**Mod** is (pre)additive, which implies by definition that its endomorphism sets are rings, with multiplication defined as composition of endomorphisms. In fact, if $x \in \mathrm{obj}(R\text{-}\mathbf{Mod})$, then $\mathrm{end}_{R\text{-}\mathbf{Mod}}(x)$ is an R-algebras and a pretracial monoidal product representation $F : \mathbf{C} \to \mathbf{Ring_{Add}}$ with respect to $\mathbf{A} = R\text{-}\mathbf{Mod}$ is a higher pretracial monoidal product representation.

LEMMA 3.1.8. Let $F : \mathbf{C}^* \to R\text{-}\mathbf{Alg}$ be a higher pretracial monoidal product representation. Then by composition with the n^{th} cyclic homology functor HC_n from Remark 3.1.3,

$$HC_n \circ F : \mathbf{C}^* \to HC_n(F(\mathbf{C}^*)) \subset \mathbf{Ab}.$$

is a monoidal product representation with insertion homomorphisms

$$\widetilde{\eta}_{\otimes y} c := HC_n(\eta_{\otimes y} c) : HC_n(F(c)) \to HC_n(F(c \otimes y))$$

and $(HC_n(F(\mathbf{C}^*)), \widetilde{\eta}_y^k)$ inherits the structure of a presimplicial set.

PROOF. The proof of Lemma 1.4.20 is based on the fact that $HC_0 = \Pi$ is a covariant functor. Therefore, the same argument works for the functors HC_n, $n > 0$.

$\square$

Higher traces of order n on an R-algebra $\mathcal{A}$ can be defined as homomorphisms from $HC_n(\mathcal{A})$ to R, i.e. $\mathbb{HT}\mathrm{race}_{[n]}(\mathcal{A}, R) := \mathrm{Hom}(HC_n(\mathcal{A}), R)$. We can obtain higher traces via cyclic cohomology in the following way.

First of all, let us consider the module $C^n_\lambda(\mathcal{A})$ of *cyclic cochains*, i.e. the submodule of $C^n(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}^{\otimes n+1}, R)$ of linear functionals $f \in C^n(\mathcal{A})$ such that $f(a_0 \otimes \cdots \otimes a_n) = (-1)^n f(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$.

Definition 3.1.9 (2.4.2, [**46**]). The homology of the complex $(C^n_\lambda(\mathcal{A}), \beta)$, with $\beta : C^n_\lambda(\mathcal{A}) \to C^{n+1}_\lambda(\mathcal{A})$ defined as:

$$\begin{aligned}\beta(f)(a_0 \otimes \cdots \otimes a_{n+1}) := &\sum_{i=0}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^{n+1} f(a_{n+1} a_0 \otimes a_1 \otimes \cdots \otimes a_n),\end{aligned}$$

is called *cyclic cohomology* and denoted by $HC^*(\mathcal{A})$. If we restrict to those functionals $f \in C^n_\lambda(\mathcal{A})$ such that $f(a_0 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some $i \in \{0, 1, \ldots, n\}$, then we obtain a subcomplex $(\overline{C}^n_\lambda(\mathcal{A}), \beta)$ whose homology $\overline{HC}^n(\mathcal{A})$ is called *reduced cyclic cohomology* (§8.3, [**45**]).

There is a *Kroneker product pairing* between cyclic homology and cohomology:

$$\langle \cdot, \cdot \rangle : HC^n(\mathcal{A}) \times HC_n(\mathcal{A}) \to R.$$

This pairing defines a map $HC^n(\mathcal{A}) \to \mathrm{Hom}(HC_n(\mathcal{A}), R) = \mathbb{HT}\mathrm{race}(\mathcal{A}, R)$, which can be an isomorphism, for example when R is a field (Remark 2.4.8, [**46**]).

Example 3.1.10 (§1.2.1, [**75**]). If $\mathcal{A}$ is a unital $\mathbb{C}$-algebra, we can identify $HC^0(\mathcal{A}) = \mathrm{Hom}(HC_0(\mathcal{A}, \mathbb{C})) = \mathrm{Trace}(\mathcal{A}, \mathbb{C})$.

Remark 3.1.11. If the unital R-algebra $\mathcal{A}$ is also Fréchet[2] and locally convex, then we can define *topological cyclic homology*, by considering completed projective tensor products (§8.6, [**45**]), and *topological cyclic cohomology*, by considering only *continuous* linear functionals.

Example 3.1.12 (§8.7, [**45**]). C^*-algebras are Fréchet algebras. For example, so are $C(M)$, $C^\infty(M)$, and $C^\infty(M, E)$. These ones are also locally convex.

Therefore, we can generalize Definition 1.4.21 to the following:

[2] An R-algebra is called *Fréchet* if it is a topological vector space for which the product is continuous.

DEFINITION 3.1.13. A symmetric monoidal category $\mathbf{C}$ has a *categorical higher trace* τ of order $n \in \mathbb{N}$ if there exist elements $c \in \mathrm{obj}(\mathbf{C})$ for which we have a non-empty subclass $\mathrm{end}^{\tau}_{\mathbf{C}}(c) \subset \mathrm{end}_{\mathbf{C}}(c)$ and a map $\tau_c : \mathrm{end}^{\tau}_{\mathbf{C}}(c) \to \mathrm{end}_{\mathbf{C}}(1_{\mathbf{C}})$ such that it pushes down to a map $\widetilde{\tau_c} : HC_n(\mathrm{end}^{\tau}_{\mathbf{C}}(c)) \to \mathrm{end}_{\mathbf{C}}(1_{\mathbf{C}})$.

EXAMPLE 3.1.14. By pairing with cyclic cohomology, R-**Mod** is a symmetric monoidal category with categorical higher traces. For example, a higher trace of order 1 $\tau_c : \mathrm{end}(c) \to \mathrm{end}(R) = R$ is a group homomorphism that restricts to $\widetilde{\tau_c} : HC_1(\mathrm{end}(c)) \to R$. In other words, it must be a R-linear morphism whose restriction to $\ker b_1 = \{a \otimes b \in C^{\lambda}_1(\mathrm{end}(c)) \mid ab - ba = 0\}$ vanishes on $\mathrm{im} b_2 = \{ab \otimes c - a \otimes bc + ca \otimes b \mid a \otimes b \otimes c \in C^{\lambda}_2(\mathrm{end}(c))\} \subseteq \ker b_1$, i.e.:

$$\tau_c(ab) \otimes \tau_c(c) - \tau_c(a) \otimes \tau_c(bc) + \tau_c(ca) \otimes \tau_c(b) = 0 \qquad \forall a \otimes b \otimes c \in C^{\lambda}_2(\mathrm{end}(c)).$$

Following Definition 1.4.23, we have:

DEFINITION 3.1.15. If in addition the background additive category $\mathbf{A}$ has a *higher F-compatible trace* τ of order n, i.e. $\forall c \in \mathrm{obj}(\mathbf{C})$, the ring homomorphism $\tau_c : Fc := \mathrm{end}^{\tau}_{\mathbf{A}}(a_c) \to \mathrm{end}_{\mathbf{A}}(1)$ satisfies:

$$\tau_{c\otimes y} \circ \eta_{\otimes y} c = \tau_c \text{ and } \tau_{c\otimes y} \circ \mu_{\otimes y} = \tau_c. \tag{3.1.1}$$

then $F : \mathbf{C}^* \to R$-**Alg** is called *higher tracial monoidal product representation* of $\mathbf{C}$ of order n.

REMARK 3.1.16. In analogy with Remark 1.4.24, from Definition 3.1.13 we have that the identities (3.1.1) push down to:

$$\widetilde{\tau}_{c\otimes y} \circ HC_n(\eta_{\otimes y} c) = \widetilde{\tau}_c \text{ and } \widetilde{\tau}_{c\otimes y} \circ HC_n(\mu_{\otimes y}) = \widetilde{\tau}_c.$$

DEFINITION 3.1.17. Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category. Recall that $HC_n(F(\mathbf{C}^*))$ has a presimplicial set structure defined by the monoidal product representation (Lemma 3.1.8), for $F : \mathbf{C}^* \to R$-**Alg** a strict higher pretracial monoidal product representation. Then a *higher logarithmic functor* of order n, or *higher log-functor* of order n, is a pre-simplicial log-additive map

$$\log_{[n]} : (\mathcal{N}\mathbf{C}, d_j, s_j) \to \left(HC_n(F(\mathbf{C}^*)), \widetilde{\eta}^j\right),$$

and is said to define a higher logarithmic representation of $\mathbf{C}$. In other words, a higher log-functor is a simplicial system on $\mathcal{N}_1\mathbf{C}$ of maps

$$\log_{[n],x\otimes y} : \mathrm{mor}(x, y) \to HC_n(F(x \otimes y)), \quad \alpha \mapsto \log_{[n],x\otimes y} \alpha, \quad x, y \in \mathrm{obj}(\mathbf{C}) \setminus 1_{\mathbf{C}}$$

such that if $\alpha \in \mathrm{mor}(x, y)$ and $\beta \in \mathrm{mor}(y, z)$, then

$$\begin{aligned}\log_{[n],x\otimes y\otimes z}(\alpha, \beta) &= \widetilde{\eta}_y(\log_{[n],x\otimes z} \beta \circ \alpha)\\ &= \widetilde{\eta}_{\otimes z}(\log_{[n],x\otimes y} \alpha) + \widetilde{\eta}_{x\otimes}(\log_{[n],y\otimes z} \beta) \in HC_n(F(x \otimes y \otimes z)).\end{aligned}$$

Remark 3.1.18. Again, it is enough to specify the maps on $\mathcal{N}_1\mathbf{C}$, i.e. to define $\log_{[n],x\otimes y}$ on $\mathrm{mor}(x, y)$ for each $x, y \in \mathrm{obj}(\mathbf{C})$. Moreover, from the definition one has again all the other properties of logarithms, e.g. the logarithm of an idempotent object is trivial.

Example 3.1.19. A log-functor is therefore a higher logarithmic functor of order 0. An example of higher logarithm of order 1, i.e. a logarithm in HC_1 can be the following. Let $Gl_1(\mathcal{A})$ be the group of invertible elements of an algebra $\mathcal{A}$ and let $a \in Gl_1(\mathcal{A})$. If we set $\log_{[1]} a := a^{-1} \otimes a$, then $b_1(a^{-1} \otimes a) = a^{-1}a - aa^{-1} = 0$ and

$$\begin{aligned}\log_{[1]} ab &= b^{-1}a^{-1} \otimes ab\\ &= b^{-1} \otimes a^{-1}ab - abb^{-1} \otimes a^{-1} + \overbrace{b^{-1}a^{-1} \otimes ab - b^{-1} \otimes a^{-1}ab + abb^{-1} \otimes a^{-1}}^{=\rho}\\ &= b^{-1} \otimes b - a \otimes a^{-1} + \rho = b^{-1} \otimes b + a^{-1} \otimes a + \rho\\ &= \log_{[1]} a + \log_{[1]} b + \rho,\end{aligned}$$

where $\rho \in \mathrm{im} b_2$ and $a^{-1} \otimes a = -a \otimes a^{-1}$ in $C_1^{\lambda}(Gl_1(\mathcal{A}))$. Hence, in $HC_1(Gl_1(\mathcal{A}))$ $\log_{[1]} ab = \log_{[1]} a + \log_{[1]} b$.

Definition 3.1.20. Let F be a higher tracial monoidal product representation of a symmetric monoidal category $\mathbf{C}$, with τ a higher trace of order n. Then the higher τ-character of the log-functor defines a *higher log-determinant functor representation* of $\mathbf{C}$ of order n. For $c \in \mathrm{obj}(\mathbf{C})$, let τ_c push down to $\widetilde{\tau}_c$ on $HC_n(F(c))$ (Remark 3.1.16). Then $\forall\ \alpha \in \mathrm{mor}_{\mathbf{C}}(c, c')$ the log-determinant functor representation is defined as $\widetilde{\tau}(\log \alpha) := \widetilde{\tau}_{c\otimes c'} \circ \log_{[n],c\otimes c'} \alpha \in \mathrm{end}_{\mathbf{A}}(1)$.

Remark 3.1.21. With the obvious generalizations of Lemma 2.19 and Lemma 2.20, **[72]**, we have once again that the log-determinant representation is independent of the insertion maps (of any order):

$$\widetilde{\tau}_{c\otimes c'}(\log_{[n],c\otimes c'} \alpha) = \widetilde{\tau}_{c\otimes c'\otimes y}(\log_{[n],c\otimes c'\otimes y} \alpha), \tag{3.1.2}$$

and that a log-determinant is independent of where it is computed:

$$\widetilde{\tau}(\log \beta\alpha) = \widetilde{\tau}(\log \alpha) + \widetilde{\tau}(\log \beta), \qquad \alpha \in \mathrm{mor}(c, c'), \quad \beta \in \mathrm{mor}(c', c'').$$

Also, as in Remark 1.4.32, a higher log-functor can be extended to elements $\delta \in \mathrm{mor}_{\mathbf{C}}(1,1)$. In fact, after choosing $\alpha \in \mathrm{mor}_{\mathbf{C}}(1,z)$ and $\beta \in \mathrm{mor}_{\mathbf{C}}(z,1)$ such that $\delta = \beta \circ \alpha$ and $z \neq 1$, we can define:

$$\log_{[n],z} \delta := \log_{[n],1\otimes z\otimes 1} (\alpha, \beta) \in HC_n(F(1 \otimes z \otimes 1)).$$

Again, it depends on δ and z (not on α and β), and if a categorical trace τ is defined, the corresponding log-determinant $\widetilde{\tau}(\log_z \delta) = \widetilde{\tau}(\log \alpha) + \widetilde{\tau}(\log \beta)$ depends only on δ.

Finally, we define higher LogTQFTs. Non trivial higher LogTQFTs arise when the cobordisms have extra structure defined on them, i.e. on specific subcategories $\mathbf{C} \subset \mathbf{Cob}_n$.

DEFINITION 3.1.22. Let $F : \mathbf{C}^* \to R\text{-}\mathbf{Alg}$ be an unoriented higher pretracial monoidal product representation of a subcategory $\mathbf{C} \subset \mathbf{Cob}_m$. Then

$$\log_{[n]} : \mathcal{N}\mathbf{C} \to HC_n(F(\mathbf{C}^*)) \tag{3.1.3}$$

is called *higher logarithmic Topological Quantum Field Theory* relative to F of dimension m and order n, or *higher LogTQFT*.

(3.1.3) corresponds to a simplicial system

$$\log_{[n],M_1\sqcup M_2} : \mathrm{mor}_{\mathbf{C}}(M_1, M_2) \to HC_*(F(M_1 \sqcup M_2))$$

and a logarithm $\log_{M_1\sqcup M_2} \overline{W}$ is identified as an element $\log_{\partial W} \overline{W} \in HC_n(F(\partial W))$, since $F(\partial W) \cong F(M_1 \sqcup M_2)$. Also, for $C_M = \overline{M \times [0,1]}$, then the proof of $\widetilde{\eta}_M \log_{M\sqcup M} C_M = 0 \in F_{\Pi}(M \sqcup M \sqcup M)$ extends to $HC_*(F(M \sqcup M \sqcup M))$ in a straightforward way.

REMARK 3.1.23. In the following chapters, we will see examples of two higher LogTQFTs, i.e. *Logarithmic Fibred QFT* (LogFQFT) and *Logarithmic Homotopy QFT* (LogHQFT), respectively defined when $\mathbf{C} = \mathbf{FCob}_m(B)$, the category of cobordisms fibred over a manifold B, or $\mathbf{C} = \mathbf{HCob}_m(B)$, the category of homotopy classes of continuous maps into a path connected space B.

3.2. Universal log-functors

In addition to the canonical projection $\pi : R \to R/[R,R]$, we also have the projection onto the algebraic K-theory group $K_0(R)$, which corresponds to a functor $K_0 : \mathbf{Ring} \to \mathbf{AbGrp}$. Since $K_0(R)$ has the universal property (Definition

3.2.2), there exists a unique abelian group homomorphism $\tau : K_0(R) \to \frac{R}{[R,R]}$ called *Hattori-Stallings trace map* (§2, Chapter II, [**86**], and §8.5.1, [**46**]) that factorizes π, i.e. the diagram

$$\begin{tikzcd} M_r(R) \arrow[r] \arrow[dr, "\pi"'] & K_0(R) \arrow[d, "\tau"] \\ & \frac{R}{[R,R]} \end{tikzcd}$$

commutes, where $M_r(R)$ is the associative ring of $r \times r$-matrices with entries in the ring R. Here, the horizontal map is the monoidal map in Definition 3.2.2.

EXAMPLE 3.2.1 (2.5.4, [**86**]). If $R = \mathbb{C}$ and $n = 1$, then π is the identity and the Hattori-Stallings trace τ corresponds to the natural inclusion of $K_0(\mathbb{C}) \cong \mathbb{Z}$ in $\mathbb{C}$.

Therefore, the idea is to define LogTQFT at the level of K-theory, thus refining the definition of higher LogTQFT. We begin with some standard definition about algebraic K-theory.

3.2.1. Algebraic K-theory and log-funtors. The definitions and results of this paragraph on algebraic K-theory are taken from Chapter II, [**86**]. We remark that the contruction is algebraic and applies to rings, but we will restrict our work to Banach algebras, and therefore the K-theory that will arise will be operator K-theory.

DEFINITION 3.2.2. Let M be an abelian monoid. The *abelian group completion* of M is an abelian group, denoted $M^{-1}M$, with a monoid map $[\cdot] : M \to M^{-1}M$ with a *universal property*, i.e. if A is an abelian group and $f : M \to A$ a monoid map, then there exists a unique group homomorphism $\widetilde{f} : M^{-1}M \to A$ such that $f = \widetilde{f} \circ [\cdot]$.

PROPOSITION 3.2.3. The group completion $M^{-1}M$ of an abelian monoid M has the following characterizing properties:

i) $M^{-1}M = \{[m] - [n] \mid m, n \in M\}$;
ii) $[m] = [n]$ in $M^{-1}M$ if and only if $m + p = n + p$, for some $p \in M$;
iii) the monoid map $(m, n) \mapsto [m] - [n]$ is surjective;
iv) $M^{-1}M$ is the set-theoretic quotient of $M \times M$ by the equivalence relation generated from $(m, n) \sim (m + p, n + p)$.

DEFINITION 3.2.4. Let $\mathbf{P}(R)$ denote the set of isomorphism classes of finitely generated projective (left) R-modules[3]. Then $(\mathbf{P}(R), \oplus)$ is an abelian monoid with identity 0_R. Hence, the *Grothendieck group* of a ring R is the abelian group completion $K_0(R) := \mathbf{P}(R)^{-1}\mathbf{P}(R)$. In particular, for $r, s \in \mathbf{P}(R)$, $[r] + [s] := [r \oplus s]$.

REMARK 3.2.5. If R is commutative, then $\mathbf{P}(R)$ is a commutative semiring with product $\otimes_R$. Consequently, $K_0(R)$ is a commutative ring with multiplicative unit $1 = [R]$. Hence, K_0 is a functor from (semi)rings to rings, and from commutative rings to commutative rings, and in particular K_0 can be seen as a functor from **Ring** to **AbGrp** (§8.2, [**46**]). In particular, for any field $\mathbb{F}$ we have $K_0(\mathbb{F}) \cong \mathbb{Z}$.

EXAMPLE 3.2.6 (Grothendieck group of vector bundles). Let X be a topological paracompact space. The space of isomorphism classes $[E]$ of complex vector bundles E over X is a commutative semiring and generates an abelian group $K^0(X)$ via the relation $[E] + [F] \sim [E \oplus F]$, $\oplus$ the Whitney sum. Then $K^0(X)$ is called *the Grothendieck group of vector bundles over* X. Since the space of (continuous) sections of a vector bundle $E \to X$ is a finitely generated projective $C(X)$-module, the Serre-Swan Theorem yields (§8.2.5, [**46**]):

$$K^0(X) \cong K_0(C(X)).$$

Notice that K_0 is covariant in $C(X)$ and thus K^0 is contravariant in X. In particular, if X is a smooth manifold, then the space of smooth sections of a vector bundle is a finitely generated projective $C^\infty(X)$-module and we have

$$K_0(C(X)) = K_0(C^\infty(X)).$$

As we will mention in §3.2.2, choosing $C^\infty(X)$, i.e. a 'smoothing' of the algebra $C(X)$, will allow the construction of another fundamental ingredient: the Chern character.

From now on, let $\mathcal{A}$ be a unital R-algebra. Since $HC_n(\mathcal{A})$ is an abelian group, we can consider the functor $\mathcal{A} \to HC_*(\mathcal{A})$ as a universal trace (thus generalizing the universal trace $\pi : \mathcal{A} \to \frac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]}$) and $K_0(\mathcal{A})$ can be seen as an 'abelianization' of $\mathcal{A}$, since it can be considered as a functor **Ring** $\to$ **AbGrp** (Remark 3.2.5). With the help of K_0, we can refine the definition of higher log-functor to a 'universal' one.

[3]A *finite dimensional free module over* R is a (left) R-module that is isomorphic to R^n for some $n \in \mathbb{N}$. A *finitely generated projective module over* R is a direct summand of a finite dimensional free module, ([**46**]).

LEMMA 3.2.7. Let $F : \mathbf{C}^* \to \mathbf{Ring_{Add}}$ be a pretracial monoidal product representation. Then by composition with $K_0 : \mathbf{Ring} \to \mathbf{AbGrp}$,

$$K_0 \circ F : \mathbf{C}^* \to K_0(F(\mathbf{C}^*)) \subset \mathbf{AbGrp}$$

is a monoidal product representation with insertion homomorphisms

$$\widetilde{\eta}_{\otimes y} c := K_0(\eta_{\otimes y} c) : K_0(F(c)) \to K_0(F(c \otimes y))$$

and $(K_0(F(\mathbf{C}^*)), \widetilde{\eta}^k_y)$ inherits the structure of a presimplicial set.

PROOF. The result follows once more by functoriality of K_0.

□

DEFINITION 3.2.8. Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category and $F : \mathbf{C}^* \to \mathbf{Ring_{Add}}$ a pretracial monoidal product representation. Recall that $K_0(F(\mathbf{C}^*))$ has a presimplicial set structure defined by the monoidal product representation (Lemma 3.2.7). Then a *universal logarithmic functor*, or *universal log-functor*, is a presimplicial log-additive map

$$u\text{-}\log : (\mathcal{N}\mathbf{C}, d_j, s_j) \to \left(K_0(F(\mathbf{C}^*)), \widetilde{\eta}^j\right),$$

which is said to define a logarithmic representation of $\mathbf{C}$. In other words, a universal log-functor is a simplicial system on $\mathcal{N}_1\mathbf{C}$ of maps

$$u\text{-}\log_{x\otimes y} : \mathrm{mor}(x, y) \to K_0(F(x \otimes y)), \quad \alpha \mapsto u\text{-}\log_{x\otimes y} \alpha, \quad x, y \in \mathrm{obj}(\mathbf{C}) \setminus 1_{\mathbf{C}}$$

such that if $\alpha \in \mathrm{mor}(x, y)$ and $\beta \in \mathrm{mor}(y, z)$, then (modulo torsion in general)

$$\begin{aligned} u\text{-}\log_{x\otimes y\otimes z}(\alpha, \beta) &= \widetilde{\eta}_y(u\text{-}\log_{x\otimes z} \beta \circ \alpha) \\ &= \widetilde{\eta}_{\otimes z}(u\text{-}\log_{x\otimes y} \alpha) + \widetilde{\eta}_{x\otimes}(u\text{-}\log_{y\otimes z} \beta) \in K_0(F(x \otimes y \otimes z)). \end{aligned}$$

If $F(c) = \mathcal{A}$ is an algebra, than the universal log-functor can yield a higher log-functor when composed with a suitable Chern character $K_0(\mathcal{A}) \to HC_*(\mathcal{A})$, which in turns can be considered as a trace , i.e. an homomorphism on the abelianization of $\mathcal{A}$ taking values into an abelian group.

3.2.2. Chern characters from the algebraic point of view. The Chern character of a vector bundle on a manifold is a very well known object used to compute K-theoretical invariants of manifolds via mapping them into de Rham cohomology. However, its construction is way more general. Here, for the sake of completeness, we recall the (non-commutative) formulation of the Chern character, as a group homomorphism $\mathrm{ch}_n : K_0 \to HC_{2n}$, which can be found in §8, [**46**].

An isomorphism class of finitely generated projective R-modules can be associated to an idempotent element e of the matrix algebra $M_r(\mathcal{A})$. Let us define

$$c(e) := (y_n, z_n, y_{n-1}, z_{n-1}, \dots, y_1) \in M_r(\mathcal{A})^{\otimes 2n+1} \oplus M_r(\mathcal{A})^{\otimes 2n} \oplus \cdots \oplus M_r(\mathcal{A}),$$

where $y_i := (-1)^i \frac{(2i)!}{i!} e^{\otimes 2i+1}$ and $z_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} e^{\otimes 2i}$.

THEOREM 3.2.9 (8.3.2, 8.3.4, [**46**]). Let $\mathcal{A}$ be a unital R-algebra (not necessarily commutative), with R a commutative ring. Let $tr : M_r(\mathcal{A})^{\otimes n} \to \mathcal{A}^{\otimes n}$ be the *generalized trace map* (§1.2.1, [**46**]). Then for any $n \in \mathbb{N}$ there are well-defined maps, functorial in $\mathcal{A}$:

$$\mathrm{ch}_n : K_0(\mathcal{A}) \to HC_{2n}(\mathcal{A}), \quad \mathrm{ch}_n([e]) := \mathrm{tr}(c(e)).$$

Hence, ch is a natural transformation $K_0 \to HC_*$ and a universal (higher) trace, taking values in the abelian groups defined as cyclic homology of the algebra. It vanishes on higher commutators (as much as the Hattori-Stallings trace vanishes on simple commutators $[r, s]$).

REMARK 3.2.10 (§8.2.6, [**46**]). This general definition reduces to the classical Chern character *à la Chern-Weil* (i.e. defined via (super-)connections, §8.1.1, [**46**]) when $\mathcal{A}$ is commutative. In particular, if $R = \mathbb{C}$ and $\mathcal{A} = C^\infty(B)$, B smooth manifold, i.e. in the case of fibre bundles, then $K_0(C^\infty(B)) \cong K^0(B)$ (Example 3.2.6), $HC_*(C^\infty(B)) \cong H^*(B, \mathbb{C})$ (by de Rham Theorem) and ch is identified with the usual ring homomorphism $K^0(B) \to H^*(B, \mathbb{C})$.

EXAMPLE 3.2.11 (8.3.6, [**46**]). For $n = 0$, $\mathrm{ch}_0 : K_0(\mathcal{A}) \to \frac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]}$ is just induced by the trace of e. If in particular $\mathcal{A}$ is a field, then $K_0(\mathcal{A}) \cong \mathbb{Z}$ and ch_0 is isomorphic to the inclusion $\mathbb{Z} \hookrightarrow \mathcal{A}$. In fact, $\mathrm{ch}_0 : K_0(\mathcal{A}) \to HC_0(\mathcal{A})$ corresonds to the Hattori-Stallings trace $\tau : K_0(\mathcal{A}) \to \frac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]}$ (Proposition 8.5.3, [**46**]).

REMARK 3.2.12. The Chern character is a natural transformation of the functors $K_0 \to HC_*$. As such, it relates in a canonical way the insertion morphisms $\widetilde{\eta}_j$ of $(K_0(F(\mathbf{C}^*)), \widetilde{\eta}_j)$ to the insertion morphisms $\widetilde{\widetilde{\eta}}_j$ of $(HC_*(F(\mathbf{C}^*)), \widetilde{\widetilde{\eta}}_j)$, i.e. $\mathrm{ch} \circ \widetilde{\eta} = \widetilde{\widetilde{\eta}}$ and thus it is possible to obtain a higher LogTQFT from a universal LogTQFT in a canonical way.

REMARK 3.2.13 (§8.7, [**45**]). Sometimes, one must require additional structure for the algebra in order to have an interesting Chern character. In fact, the topological cyclic homology and cohomology of a C^*-algebra can be quite poor. For instance, for a manifold M, $HC^n(C(M)) = HC^0(C(M))$ if n is even and $HC^n(C(M)) = 0$ when n is odd. Therefore, when dealing with a C^*-algebra $\mathcal{A}$, it is usually better to consider a *smooth subalgebra* $\mathcal{B} \subset \mathcal{A}$, i.e. a Fréchet locally convex dense subalgebra closed under holomorphic functional calculus, such as $C^\infty(M) \subset C(M)$, in order to have an interesting cyclic homology and thus an interesting Chern character. This choice does not alter the K-theory involved, since there exists a canonical isomorphism identifying $K_0(\mathcal{B}) = K_0(\mathcal{A})$.

3.2.3. Morita Equivalence.

DEFINITION 3.2.14 (Definition 1.2.5, [**46**]). Two unital R-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* if there is an $\mathcal{A}$-$\mathcal{B}$-bimodule P, an $\mathcal{B}$-$\mathcal{A}$-bimodule Q, an isomorphism of $\mathcal{A}$-bimodules $u : P \otimes_{\mathcal{B}} Q \cong \mathcal{A}$ and an isomorphism of $\mathcal{B}$-bimodules $v : Q \otimes_{\mathcal{A}} P \cong \mathcal{B}$.

THEOREM 3.2.15 (2.2.9 & 2.4.6, [**46**]). Let $\mathcal{A}$ and $\mathcal{B}$ two Morita equivalent unital or H-unital[4] R-algebras. Then there exist canonical isomorphisms such that $HC_*(\mathcal{A}) \cong HC_*(\mathcal{B})$ and $HC^*(\mathcal{A}) \cong HC^*(\mathcal{B})$.

EXAMPLE 3.2.16. We have already seen that the tracial monoidal product representation $F_{-\infty} : \mathbf{Cob}^*_{2n} \to \mathbb{C}\text{-}\mathbf{Alg}$, $F_{-\infty}(M) = \Psi^{-\infty}(M,E)$ allows us to define a LogTQFT $\log_M \overline{W} := \pi_*(\kappa_\sharp(\mathcal{C} - \mathcal{P})) \in \Pi \circ F_{-\infty}(M)$ with trace character $\widetilde{\mathrm{Tr}}(\log_M \overline{W}) = \mathrm{ind}(\mathcal{PC}) \in \mathbb{Z}$. But we also have that $[\mathcal{C} - \mathcal{P}] = \mathrm{ind}(\mathcal{PC}) \in \mathbb{Z} = K_0(\mathbb{C})$. In fact, $\Psi^{-\infty}(M,E)$ is a C^*-algebra, and as such it is H-unital (see for instance [**87**]). In particular, by Schwarz's Kernel Theorem, $\Psi^{-\infty}(M,E) \cong C^\infty(M \times M, \mathrm{End}(E))$, and hence it is Morita equivalent[5] to $\mathrm{End}(E) \cong \mathrm{End}(\mathbb{C}^N)$, which in turn is Morita equivalent to $\mathbb{C}$. Therefore $K_0(F_{-\infty}(M)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$, canonically, and we can define a universal LogTQFT as $\widetilde{\log}_M \overline{W} := [\mathcal{C} - \mathcal{P}]$, whose log-character $\tau(\widetilde{\log}_M \overline{W}) \in \mathbb{C}$ is obtained via $\tau = \mathrm{ch}_0$, the Hattori-Stallings trace.

REMARK 3.2.17. Morita equivalence provides an alternative proof that:

$$HC_0(\Psi^{-\infty}(M,E)) = \frac{F_{-\infty}(M)}{[F_{-\infty}(M), F_{-\infty}(M)]} \cong \mathbb{C}$$

which is shown in Lemma 2.3, [**72**], via the unique (classical) trace Tr on smoothing pseudodifferential operators.

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[4] *Homologically unital*. For a definition, see [**87**]. For example, all C^*-algebras are H-unital.

[5] See §1, [**51**].

CHAPTER 4

LogTQFT for families

In this chapter we extend the results of [**72**] on topological signature and LogTQFT to the family signature and LogFQFT. The key point is represented by the fact that EBVPs have a family counterpart, made of *families of realizations*, which have a well defined index, now as a class in $K^0(B)$.

Boundary conditions are represented by *spectral sections*, among which we need, for the family signature, *symmetric* ones ([**42**]), which will provide a homotopy invariant index. They extend to families the concept of generalized APS condition that we mentioned in Chapter 2.

4.1. Fibre bundles and their bordism groups

Let $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$ denote a fibre bundle, i.e. a smooth surjective surjection[1] onto a closed manifold B. We will call $\mathcal{X}$ the *total space*, B the *base*, and X the *fibre* of the fibre bundle. When X is closed, the structure group of the fibre bundle is $\mathrm{Diff}(X)$, the group of diffeomorphisms of the fibre X, while if $Y := \partial X \neq \emptyset$ then the structure group of $X \hookrightarrow \mathcal{X} \to B$ is $\mathrm{Diff}(X, Y)$, the group of diffeomorphisms of X that leave the boundary Y invariant (§3, [**13**]).

We will be interested in families of cobordisms, i.e. *fibred cobordisms*, and therefore we investigate the relationship between fibre bundles with closed fibre and those whose fibre has a boundary.

PROPOSITION 4.1.1. Let $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$ be a fibre bundle and $Y := \partial X \neq \emptyset$. Then there exists a fibre bundle $Y \hookrightarrow \mathcal{Y} \xrightarrow{\rho} B$ such that $\mathcal{Y} = \partial\mathcal{X}$ and $\pi_{|\mathcal{Y}} = \rho$.

PROOF. Consider the structure group $\mathrm{Diff}(X, Y)$. Then, by composing with the inclusion $Y \hookrightarrow X$, we obtain well defined transition maps for Y, which in turns define the bundle $Y \hookrightarrow \mathcal{Y} \xrightarrow{\pi} B$ with the desired properties.

□

REMARK 4.1.2. The converse of Propositions 4.1.1 needs not to be true and a counterexample is provided at the beginning of [**17**]. In fact, if $Y \hookrightarrow \mathcal{Y} \xrightarrow{\pi} B$ is a

[1] Equivalently, we refer to Definition 1.1, [**8**]

fibre bundle and $Y = \partial X$, then it is not necessarily true that there exists a fibre bundle with fibre X such that $\mathcal{Y}$ is its boundary. This is because $\mathrm{Diff}(Y)$ needs not to refine to those diffeomorphisms that extend to X and leave Y invariant.

Hence, it is not enough to assume that our fibre bundle has a bording fibre for the total space $\mathcal{Y}$ to bord. Unfortunately, even when there exists a manifold $\mathcal{X}$ such that $\partial\mathcal{X} = \mathcal{Y}$, it is not straightforward that $\mathcal{X}$ is a fibre bundle, at least on the same base space. In fact, let S^1 be the unit circle considered as a fibre bundle over itself, the fibre being a point and the bundle map being the identity. Clearly, S^1 bounds the unit disc D, but D is not a fibre bundle over S^1, as its Euler characteristic does not vanish modulo 2 (see [**17**] for the use of the mod 2 Euler characteristic in determining those manifolds that can be fibred over S^1). Equivalently, there are no continuous functions $D \to S^1$ that are the identity on $S^1 = \partial D$.

Let $X_i \hookrightarrow \mathcal{X}_i \xrightarrow{\pi_i} B$, $i = 1, 2$, be two fibre bundles over B. A *fibre bundle morphism* is a smooth map $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\pi_1 = \pi_2 \circ \varphi$. Moreover, if φ is a diffeomorphism, then it is called a *fibre bundle isomorphism*.

REMARK 4.1.3 (Chapter 2, [**33**]). Since $\varphi(\pi_1^{-1}(b)) \subseteq \pi_2^{-1}(b)$ $\forall b \in B$, then the fibres are automatically preserved when φ is a fibre bundle isomorphism, i.e. φ restricts to a diffeomorphism $X_1 \to X_2$.

DEFINITION 4.1.4. We denote by $\mathbf{FDiff}(B)$ the category of fibre bundles over B as objects and fibre bundle isomorphisms as arrows. When endowed with disjoint union of Definition 4.1.9, it becomes a symmetric monoidal category and a subcategory of $\mathbf{Diff}(\{\mathrm{pt}\}) =: \mathbf{Diff}$, the category of manifolds and diffeomorphisms between them. Let it be denoted $\mathbf{FDiff}_n(B)$ when the total space has dimension n. Again, $\mathbf{FDiff}_n(B)$ is a symmetric monoidal category when considered together with disjoint union.

Let $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$ a fibre bundle with boundary bundle $Y \hookrightarrow \mathcal{Y} \xrightarrow{\pi'} B$, and let $E := E_b$ be a smooth vector bundle over $X = X_b$, $b \in B$. For $E' := E_{|Y}$, let $\mathrm{Diff}(X, E; E')$ denote the subgroup of $\mathrm{Diff}(E)$ of diffeomorphisms mapping linearly fibres into fibres and preserving E'. When $\mathcal{Y} = \emptyset$, we denote it by $\mathrm{Diff}(X, E)$. Then $\mathrm{Diff}(X, E; E')$, respectively $\mathrm{Diff}(X, E)$, is a topological group and a subgroup of $\mathrm{Diff}(E)$ ((1.1) in [**5**], and §3 in [**13**]).

DEFINITION 4.1.5 (§2.1, [**69**]). A *smooth family of vector bundles* associated to $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$ is a finite rank smooth vector bundle $\mathcal{E} \xrightarrow{\rho} \mathcal{X}$. Hence, the composition $\mathcal{E} \xrightarrow{\pi\circ\rho} B$ is a fibre bundle with fibre $E_b := E_{|X_b}$ and structure group $\mathrm{Diff}(X, E; E')$.

EXAMPLE 4.1.6 (§2.1, **[69]**). Fundamental examples of smooth families of vector bundles are the tangent and cotangent bundles of $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$, i.e. $T\mathcal{X}$ and $T^*\mathcal{X}$, respectively. In the sequel, we will consider the *vertical tangent bundle*, i.e. the sub-bundle $T(\mathcal{X}/B) := T_\pi\mathcal{X} := \bigcup_{b\in B} T_b X$ of $T\mathcal{X}$. Likewise, we have the dual vertical bundle $T^*(\mathcal{X}/B) := T^*_\pi\mathcal{X} := \bigcup_{b\in B} T^*_b X$ of vertical differential 1-forms and the pull-back bundles π^*TB and π^*T^*B from the base.

REMARK 4.1.7 (§2.1, **[69]**). If $E \hookrightarrow \mathcal{E} \to B$ is a family of vector bundles, then there is an infinite-dimensional smooth Fréchet bundle $\pi_*(E) \hookrightarrow \pi_*(\mathcal{E}) \to B$ associated to it, with fibre $\pi_*(E) := \pi_*(E_b) = C^\infty(X_b, E_b)$, $\forall b \in B$. The space of sections of $\pi_*(\mathcal{E})$ is $C^\infty(B, \pi_*(\mathcal{E}))$ and corresponds to $C^\infty(\mathcal{X}, \mathcal{E})$, a $C^\infty(B)$-module. In practice, one works with the right hand side.

In general, we have the de Rham complex of smooth forms on B with values in $\pi_*(\mathcal{E})$, i.e. the graded algebra $\mathcal{A}(B, \pi_*(\mathcal{E})) = \bigoplus_{k=0}^{\dim X} \mathcal{A}^k(B, \pi_*(\mathcal{E}))$ where:

$$\mathcal{A}^k(B, \pi_*(\mathcal{E})) := C^\infty(\mathcal{X}, \pi^*\Lambda^k(B) \otimes \mathcal{E}).$$

REMARK 4.1.8. Let $d\pi : T\mathcal{X} \to TB$ be the differential of π. Then $T_\pi\mathcal{X} = \ker d\pi$ and it fits in the short exact sequence:

$$0 \longrightarrow \ker \pi_* \longrightarrow T\mathcal{X} \longrightarrow \pi^*TB \longrightarrow 0,$$

where π^*TB is the pull-back bundle of $TB \to B$. Then a connection corresponds to a splitting of the sequence and therefore to a sub-bundle $T_H\mathcal{X} \cong \pi^*TB$ which complements $T_\pi\mathcal{X}$, i.e.

$$T\mathcal{X} \cong T_\pi\mathcal{X} \oplus T_H\mathcal{X} \cong T_\pi\mathcal{X} \oplus \pi^*TB.$$

From now on, $X \hookrightarrow \mathcal{X} \xrightarrow{\pi} B$ will also be denoted $(\mathcal{X}, \pi)$, if we do not need to specify the fibre.

DEFINITION 4.1.9. Let $(\mathcal{X}, \pi)$ and $(\mathcal{W}, \rho)$ be fibre bundles over B and fibres X and W, respectively. Then we define:

i) *inverse orientation* as the fibre bundle $(\mathcal{X}, \pi)^- := (\mathcal{X}^-, \pi)$ with fibre X^-;
ii) *disjoint union* as the fibre bundle $(\mathcal{X}, \pi) \sqcup (\mathcal{W}, \rho) := (\mathcal{X} \sqcup \mathcal{W}, \pi \sqcup \rho)$ with fibre $X \sqcup W$, where:

$$(\pi \sqcup \rho)_{|\mathcal{X}} = \pi \text{ and } (\pi \sqcup \rho)_{|\mathcal{W}} = \rho.$$

REMARK 4.1.10 ([**66**]). $(\mathcal{X},\pi)$, $(\mathcal{X}\sqcup\emptyset,\pi)$ and $(\emptyset\sqcup\mathcal{X},\pi)$ are not identified, but naturally diffeomorphic via the diffeomorphisms:

$$l_{(\mathcal{X},\pi)}:(\mathcal{X}\sqcup\emptyset,\pi)\to(\mathcal{X},\pi)\quad\text{ and }\quad r_{(\mathcal{X},\pi)}:(\emptyset\sqcup\mathcal{X},\pi)\to(\mathcal{X},\pi).$$

DEFINITION 4.1.11. Let $Y\hookrightarrow\mathcal{Y}\xrightarrow{\pi}B$ be a fibre bundle with Y closed. Then:

$$(\mathcal{Y},\pi)\ \textit{bords}\iff\text{there exists }(\mathcal{X},\rho)\text{ such that }\partial\mathcal{X}=\mathcal{Y}\text{ and }\rho_{|\mathcal{Y}}=\pi.$$

Hence $(\mathcal{Y}_i,\pi_i)$, $i=1,2$, are *bordant* if and only if $(\mathcal{Y}_1^-\sqcup\mathcal{Y}_2,\pi_1\sqcup\pi_2)$ bords. If $(\mathcal{Y}_1^-\sqcup\mathcal{Y}_2,\pi_1\sqcup\pi_2)$ bords $(\mathcal{X},\rho)$, then the latter is called *fibred cobordism* from $(\mathcal{Y}_1,\pi_1)$ to $(\mathcal{Y}_2,\pi_2)$.

Then one can show as for the single manifold case that:

PROPOSITION 4.1.12. Bordism of fibre bundles is an equivalence relation.

In the spirit of [**16**], $[\mathcal{Y},\pi]$ will denote the bordism class of a fibre bundle $(\mathcal{Y},\pi)$, in the sense of Definition 4.1.11. Then

$$\Omega_n(B):=\{[\mathcal{Y},\pi]\mid(\mathcal{Y},\pi)\text{ has closed }n\text{-dimensional fiber}\}$$

is an abelian group, the addition being $[\mathcal{Y}_1,\pi_1]+[\mathcal{Y}_2,\pi_2]:=[\mathcal{Y}_1\sqcup\mathcal{Y}_2,\pi_1\sqcup\pi_2]$. We will call it *fibred n-bordism group of B*. Finally, $\Omega_*(B)=\bigoplus_n\Omega_n(B)$ is a graded module over the Thom ring, with product:

$$[\mathcal{Y},\pi][Z]:=[\mathcal{Y}\times Z,\rho],\tag{4.1.1}$$

where $[Z]\in\Omega_*$ and $\mathcal{Y}\times Z\xrightarrow{\rho}B$ is the fibre bundle with $\eta(y,z)=\pi(y)$ $\forall y\in\mathcal{Y}$, $\forall z\in Z$ and fibre $Y\times Z$. If orientation is neglected, then we obtain $\mathfrak{N}_n(B)$, the group of equivalence classes $[\mathcal{Y},\pi]_2$ of *unoriented* fibre bundles $(\mathcal{Y},\pi)$ (the 2 clarly stands for the coefficient ring $\mathbb{Z}_2$), and the graded $\mathfrak{N}_*$-module $\mathfrak{N}_*(B)=\bigoplus_n\mathfrak{N}_n(B)$. We remark that the difference between our case and [**16**] lies in the refinement to fibre bundles.

4.2. Families of logTQFTs

We can define composition of fibred cobordisms by fibrewise gluing. To this purpose, we need a 'fibred' version of the Smooth Collaring Theorem:

PROPOSITION 4.2.1 (Proposition 4.1, [**13**]). Let $Y\hookrightarrow\mathcal{Y}\to B$ be the boundary of $X\hookrightarrow\mathcal{X}\to B$ and $\mathcal{U}$ be a sub-bundle of $\mathcal{X}$ with the open set $U\subset X$ as fibre and structure group $\mathrm{Diff}(\overline{U},Y)$, $\overline{U}$ being the closure of U. Then there exists a fibre bundle isomorphism $\Phi : [0,1) \times \mathcal{Y} \to \mathcal{U}$ which restricts to a collar neighbourhood of the boundary on each fibre.

From Definition 4.1.4, we have the symmetric monoidal category of fibre bundles over B and fibre bundles isomorphisms $\mathbf{FDiff}(B) = \bigcup_n \mathbf{FDiff}_n(B)$. We assume from now on oriented fibres and fibrewise orientation preserving diffeomorphisms. As we said, we can glue fibre bundles together into a new fibre bundle whose smooth structure depends on the choice of smooth collar. Hence, gluing is associative modulo fibre bundle isomorphism, as for the 'single' cobordism case.

Definition 4.2.2. Fibre bundles with $(n-1)$-dimensional closed fibre and fibred cobordisms between them define the category (enriched over categories) $\mathbf{FCob}_n(B)$ of cobordims fibred over B with fibre dimension n. Together with disjoint union, it is a symmetric monoidal category whose objects are fibre bundles over B with $(n-1)$-dimensional closed fibre and whose morphisms are (compositions of) fibred cobordisms over B and oriented fibre bundle isomorphisms.

Remark 4.2.3. Once gluing is defined, $\mathbf{FCob}_n(B)$ is defined as in **[80]** for the *Riemannian (co)bordism category* (the latter is more complicated because the Riemannian structure is prescribed before hand and two fibered manifolds can be glued only if their metrics coincide in a collar neighborhood of the common boundary). In fact, it arises as a category *internal* to the 2-category of symmetric monoidal categories, as $\mathbf{Cob}_n$. Since we do not aim at a precise description of such categories, we simply refer to **[80]** for the definition of categories internal to the category of strict symmetric monoidal categories and for the commuting diagrams they satisfy. Equivalently, $\mathbf{FCob}_n(B)$ can be obtained by the construction described in **[66]** for $\mathbf{HCob}_n(B)$, the category of homotopy cobordisms (which will be described briefly in Chapter 5), which is still based on the concept of categories enriched over categories.

In a similar fashion, we can consider the category of vector bundles over B, where the fibers are vector spaces over a field $\mathbb{F}$, and vector bundles morphisms between them $\mathbf{Vect}_{\mathbb{F}}(B)$ (as Definition 2.47, **[80]**, where the vector bundles are also *topological*). Then, reading off Definition 2.48 of **[80]** in our setting, we have:

Definition 4.2.4. A *Fibered Topological Quantum Field Theory* (FQFT) of dimension m over B is a symmetric monoidal functor:

$$\mathcal{Z} : \mathbf{FCob}_m(B) \to \mathbf{Vect}_{\mathbb{F}}(B).$$

Since $\mathbf{FCob}_m(B) \subseteq \mathbf{FCob}_m(\{\mathrm{pt}\}) = \mathbf{Cob}_m$, it can also be used to define a special kind of higher LogTQFT:

DEFINITION 4.2.5. A family of LogTQFTs of dimension m, or *LogFQFT*, is a higher log-functor over $\mathbf{FCob}_m(B)$, i.e.:

$$\log : \mathcal{N}\mathbf{FCob}_m(B) \to HC_*(F(\mathbf{FCob}_n(B)^*)).$$

4.3. Families of Dirac operators and boundary value problems

As in §0.2, let $\Psi^m(X; E, F)$ denote the space of classical pseudodifferential operators $A : C^\infty(X, E) \to C^\infty(X, F)$ of order m and let $\mathsf{CS}^m(X; E, F)$ be the space of classical symbols. Let $\sigma : \Psi^m(X; E, F) \to \mathsf{CS}^m(X; E, F)$ be the symbol map.

PROPOSITION 4.3.1 (§1, [**5**]). For $\mathcal{E}, \mathcal{F}$ two smooth vector bundles over $\mathcal{X}$, with fibres E and F, respectively, there is a well defined smooth family of vector bundles $\Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}) \to B$ with fibre $\Psi^m(X; E, F) := \Psi^m(X_b; E_b, F_b)$ and structure group $\mathrm{Diff}(E, F; X)$, the subgroup of $\mathrm{Diff}(E \oplus F; X)$ of diffeomorphisms mapping E to E and F to F.

Also, since σ is invariant under the action of $\mathrm{Diff}(E, F; X)$, there is a symbol bundle $\mathsf{CS}^m(X; E, F) \hookrightarrow \mathsf{CS}^m(\mathcal{X}; \mathcal{E}, \mathcal{F}) \to B$, with structure group $\mathrm{Diff}(E, F; X)$. Thus, in every local trivialization a continuous section of $\mathsf{CS}^m(\mathcal{X}; \mathcal{E}, \mathcal{F})$ is a family of symbols in $\mathsf{CS}^m(X; E, F)$, which is called a *vertical symbol*, since its cotangent variable belongs to the cotangent bundle along the fibres $T^*(\mathcal{X}/B)$.

DEFINITION 4.3.2 (§1, [**5**]). A *smooth family of ψdos of order m* associated to a fibre bundle $\mathcal{X}$ is a smooth section $\mathcal{T} \in C^\infty(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}))$. Concretely (see [**69**]), it consists of a classical ψdo $\mathcal{T} : C^\infty(\mathcal{X}, \mathcal{E}) \to C^\infty(\mathcal{X}, \mathcal{F})$ with Schwarz kernel $\kappa_{\mathcal{T}} \in \mathcal{D}'(\mathcal{X} \times_\pi \mathcal{X}, \mathcal{F} \boxtimes \mathcal{E}^*)$, such that in any local trivialization $\kappa_{\mathcal{T}}$ is an oscillatory integral whose symbol is a vertical symbol. $\mathcal{T}$ will also be called *vertical* ψdo, and we will write

$$\Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F}) = C^\infty(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}))$$

for the algebra of vertical ψdos. $\mathcal{T}$ may sometimes be denoted[2] by $\mathcal{T} = (T_b)_{b \in B}$. If T_b is elliptic $\forall b \in B$, then $\mathcal{T}$ is called *elliptic*.

EXAMPLE 4.3.3 ([**5**]). When $\mathcal{X} = B \times X$, $\mathcal{E} = B \times E$ and $\mathcal{F} = B \times F$, then $\mathcal{T}$ is just a continuous map $B \to \Psi^m(X; E, F)$. All continuous families are locally of this form.

[2]In fact, in a local trivialization $\mathcal{T}$ is identified with $T_b : C^\infty(X_b, E_b) \to C^\infty(X_b, F_b)$.

If $Y := \partial X \neq \emptyset$, i.e. $X \hookrightarrow \mathcal{X} \to B$ has a boundary $Y \hookrightarrow \mathcal{Y} \to B$, then $\mathcal{T} \in \Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F})$ is defined as $\mathcal{T} := r^+ \widetilde{\mathcal{T}} e^+$, where:

- $\widetilde{X} \hookrightarrow \widetilde{\mathcal{X}} \xrightarrow{\widetilde{\pi}} B$ is a fibre bundle with closed fibre such that $\mathcal{X} \subset \widetilde{\mathcal{X}}$ and $\widetilde{\pi}_{|\mathcal{X}} = \pi$;
- $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}} \to B$ are smooth families of vector bundles such that $\mathcal{E} = \widetilde{\mathcal{E}}_{|\mathcal{X}}$ and $\mathcal{F} = \widetilde{\mathcal{F}}_{|\mathcal{X}}$;
- $r^+ : C^\infty(\widetilde{\mathcal{X}}, \widetilde{\mathcal{F}}) \to C^\infty(\mathcal{X}, \mathcal{F})$ and $e^+ : C^\infty(\mathcal{X} \setminus \mathcal{Y}, \mathcal{E}) \to C^\infty(\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}})$;
- $\widetilde{\mathcal{T}} \in \Psi^m_{\mathrm{vert}}(\widetilde{\mathcal{X}}; \widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$;
- $\mathcal{T}$ satisfies *transmission conditions* at the boundary fibre bundle (see Chapter 0 and [**69**] for more on this).

REMARK 4.3.4 (§1, [**62**]). Following Remark 4.1.7, there is complex of smooth forms on B with values in $\Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F})$,

$$\mathcal{A}(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F})),$$

i.e. the algebra of vertical classical pseudodifferential operators with differential form coefficients. If $\mathcal{Q} \in \mathcal{A}(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}))$, then its form degree zero component $\mathcal{Q}_{[0]} \in \mathcal{A}^0(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F}))$ is a vertical ψdo; in fact,

$$\mathcal{A}^0(B, \pi_*(\mathcal{E})) = C^\infty(\mathcal{X}, \pi^*\Lambda^0(B) \otimes \mathcal{E}) = C^\infty(\mathcal{X}, \mathcal{E}) = C^\infty(B, \pi_*(\mathcal{E})).$$

Thence, $\mathcal{A}^0(B, \Psi^m(\mathcal{X}; \mathcal{E}, \mathcal{F})) = \Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F})$ and $\mathcal{Q} \in \Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F})$ if and only if $\mathcal{Q} = \mathcal{Q}_{[0]}$.

DEFINITION 4.3.5 (Definition 1, [**62**]). $\mathcal{Q} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ is elliptic, resp. admissible[3], resp. invertible, if $\mathcal{Q}_{[0],b}$ is elliptic, resp. admissible, resp. invertible, $\forall b \in B$.

If $\mathcal{T} \in \Psi^m_{\mathrm{vert}}(\mathcal{X}; \mathcal{E}, \mathcal{F})$ is elliptic and $\partial\mathcal{X} = \emptyset$, then each σ^{T_b} is invertible outside the zero section and hence each T_b is Fredholm. Hence, by Proposition 2.2 of [**5**], there exist k sections $w_1, \ldots, w_k \in C^\infty(\mathcal{X}, \mathcal{F})$ such that the map

$$\widehat{\mathcal{T}} : C^\infty(\mathcal{X}, \mathcal{E}) \oplus \mathbb{C}^k \to C^\infty(\mathcal{X}, \mathcal{F}), \qquad \widehat{T}_b(u, \lambda_1, \ldots, \lambda_k) := T_b(u) + \sum_{i=1}^k \lambda_i w_i(b),$$

is surjective $\forall b \in B$. This implies that the vector spaces $\ker(\widehat{T}_b)$ then form a vector bundle $\ker(\widehat{\mathcal{T}})$ over B and that the element $[\ker(\widehat{\mathcal{T}})] - [B \times \mathbb{C}^k] \in K^0(B)$ does not depend on the choice of the sections w_i. This yields the following:

[3] That is, there exists a spectral cut θ for the operator, i.e. its spectrum is not dense.

DEFINITION 4.3.6 ((2.3), [**5**]). The index of the elliptic family $\mathcal{T}$ is defined as:

$$\mathrm{ind}(\mathcal{T}) := [\ker \widetilde{\mathcal{T}}] - [B \times \mathbb{C}^k] \in K^0(B).$$

If in addition $\dim \ker(T_b)$ is locally constant, then the families $(\ker(T_b))_{b \in B}$ and $(\mathrm{coker}(T_b))_{b \in B}$ form the vector bundles $\ker(\mathcal{T})$ and $\mathrm{coker}(\mathcal{T})$ over B and the index of $\mathcal{T}$ can be defined as $\mathrm{ind}(\mathcal{T}) = [\ker(\mathcal{T})] - [\mathrm{coker}(\mathcal{T})] \in K^0(B)$.

REMARK 4.3.7 (§2, [**5**]). $\mathrm{ind}\mathcal{T} \in K^0(B)$ is a homotopy invariant and depends only on the homotopy class of $\sigma^{\mathcal{T}}$.

EXAMPLE 4.3.8 (§4, [**3**]). Let $\mathscr{D}^{\mathrm{Sign}} := \{\eth_b^{\mathrm{Sign}}\}_{b \in B}$ be a smooth family of signature operators, i.e. $\eth_b^{\mathrm{Sign}} : \Omega^+(X_b) \to \Omega^-(X_b)$ (where clearly the splitting is induced by the family of Hodge operators). If $\partial X = \emptyset$, then $\ker(\eth_b^{\mathrm{Sign}})$ has constant dimension and $\mathrm{ind}(\mathscr{D}^{\mathrm{Sign}}) = [\ker(\mathscr{D}^{\mathrm{Sign}})] - [\mathrm{coker}(\mathscr{D}^{\mathrm{Sign}})] \in K^0(B)$. By Atiyah-Singer Family Index Theorem, its Chern character is

$$\mathrm{ch}(\mathrm{ind}(\mathscr{D})) = \int_{\mathcal{X}/B} L(T_\pi \mathcal{X}) \in H^*(B),$$

where the map $\int_{\mathcal{X}/B} : H^*(\mathcal{X}) \to H^{*-n}(B)$ is the *integration along the fiber* (or *Gysin map*, Definition 1.5.10, [**75**] - see Proposition 6.14.1, [**11**] for the definition on cohomology). Here, $n = \dim X$.

Analogously, we obtain a smooth family of signature operators when $\partial X \neq \emptyset$, but in order to have a well-defined virtual index bundle, one has to impose suitable boundary conditions. Hopefully, the technology of Chapter 2 generalizes to the case of fibre bundles in a natural way.

Let us consider $\mathcal{X} \xrightarrow{\pi} B$ with oriented even-dimensional fibre X and boundary $Y \hookrightarrow \mathcal{Y} \xrightarrow{\pi'} B$. We consider a Riemannian metric $g^{\mathcal{X}/B}$ on $T(\mathcal{X}/B)$, thus inducing a metric $g^{\mathcal{Y}/B}$ on $T(\mathcal{Y}/B)$, such that it is of product form on a collar fibration $\mathcal{U} \to B$ (which exists by Proposition 4.2.1), i.e. $g^{\mathcal{X}/B}_{|\mathcal{U}} = dt^2 + g^{\mathcal{Y}/B}$. Let $\mathcal{E} \to \mathcal{X}$ be a family of Clifford bundles with metric $g^{\mathcal{E}}$ and unitary connection $\nabla^{\mathcal{E}}$ such that $g^{\mathcal{E}}_{|\mathcal{U}}$ and $\nabla^{\mathcal{E}}_{|\mathcal{U}}$ are both independent of t, the normal coordinate. In this way, we obtain a family $\mathscr{D} = (\eth_b)_{b \in B} \in \Psi^1_{\mathrm{vert}}(\mathcal{X}, \mathcal{E})$ of Dirac operators.

The fibrewise restriction defines a global trace map $\gamma : C^\infty(\mathcal{X}, \mathcal{E}) \to C^\infty(\mathcal{Y}, \mathcal{E}')$ corresponding to the restriction $\gamma : C^\infty(B, \pi^*(\mathcal{E})) \to C^\infty(B, \pi^*(\mathcal{E}'))$. Hence, by product structure, $\mathcal{E}_{|\mathcal{U}} = \gamma^* \mathcal{E}'$ and $C^\infty(\mathcal{U}, \mathcal{E}_{|\mathcal{U}}) = C^\infty([0,1)) \otimes C^\infty(\mathcal{Y}, \mathcal{E}')$, for $\mathcal{U}$ a fibred neighbourhood of the boundary bundle $\mathcal{Y}$. There, a family of Dirac operators

$\mathscr{D}$ decomposes as

$$\mathscr{D}_{|\mathcal{U}} = \Upsilon\left(\partial_t + \mathscr{D}_{\mathcal{Y}}\right), \tag{4.3.1}$$

where $\mathscr{D}_{\mathcal{Y}} \in \Psi^1_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ is a family of Dirac operators associated to the boundary fibration, and $\Upsilon \in C^\infty(\mathcal{Y}, \mathrm{End}(\mathcal{E}'))$ is a bundle isomorphism given by the fibrewise Clifford product by the inward unit normal to Y.

Let $\widetilde{\mathscr{D}} : C^\infty(\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}}) \to C^\infty(\widetilde{\mathcal{X}}, \widetilde{\mathcal{E}})$ be an invertible family such that $\widetilde{\mathscr{D}}_{|\mathcal{X}} = \mathscr{D}$. Then we can define a vertical Poisson operator $\mathscr{K} : C^\infty(\mathcal{Y}, \mathcal{E}') \to C^\infty(\mathcal{X}, \mathcal{E})$ and a vertical Calderón projector $\mathscr{C} := \gamma\mathscr{K} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ in the expected way. In fact, $\ker\mathscr{D} = \{s \in C^\infty(\mathcal{X}, \mathcal{E}) |\ \mathscr{D}s = 0 \text{ in } \mathcal{X} \setminus \mathcal{Y}\}$ and $\mathrm{ran}\mathscr{C} = \gamma\ker\mathscr{D}$, the space of *vertical* Cauchy data, are well defined smooth bundles (Proposition 2.1, [**70**]). Also, by fibrewise Unique Continuation, $\gamma : \ker\mathscr{D} \xrightarrow{\cong} \mathrm{ran}\mathscr{C}$ is an isomorphism, with the vertical Poisson operator $\mathscr{K}$ as a left inverse.

Unfortunately, $(\Pi_{\geq 0,b})_{b\in B}$ defines a smooth family if and only if $\dim\ker(\mathscr{D}_{\mathcal{Y}})_b$ is constant over B. Therefore, boundary conditions for families requires the more general notion of *spectral section*.

DEFINITION 4.3.9 (Definition 2.1, [**19**]). A *spectral section* $\mathscr{P}$ of $\mathscr{D}_{\mathcal{Y}}$ is a smooth family $(P_b)_{b\in B} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ of self-adjoint pseudodifferential projections of degree zero such that P_b is a finite rank perturbation of $\Pi_b := \Pi_{\geq,b}$ for each $b \in B$. In particular, all spectral section have the same principal symbol.

A *generalized spectral section*[4] $\mathscr{P}$ of $\mathscr{D}_{\mathcal{Y}}$ is a smooth family $(P_b)_{b\in B} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ of self-adjoint pseudodifferential projections such that its principal symbol is the same as that of a spectral section of of $\mathscr{D}_{\mathcal{Y}}$.

REMARK 4.3.10. The family of Calderón projectors $\mathscr{C}$ defined above is a generalized spectral section of $\mathscr{D}_{\mathcal{Y}}$ (as pointed out in [**19**]), but is a classical spectral section if $\Pi_b - C_b$ is a finite rank perturbation, e.g. when the fibre X is compact and has a product structure near the boundary.

THEOREM 4.3.11 ([**19**]). Let $\mathscr{P}_i \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$, $i = 1, 2, 3$, be generalized spectral sections of $\mathscr{D}_{\mathcal{Y}}$. Then $\mathscr{P}_2\mathscr{P}_1 := (P_{2,b}P_{1,b} : \mathrm{ran}(P_{1,b}) \to \mathrm{ran}(P_{2,b}))_{b\in B}$ is Fredholm, $\mathrm{ind}(\mathscr{P}_2\mathscr{P}_1) = [\mathscr{P}_1 - \mathscr{P}_2] \in K^0(B)$, and

$$[\mathscr{P}_1 - \mathscr{P}_2] + [\mathscr{P}_2 - \mathscr{P}_3] = [\mathscr{P}_1 - \mathscr{P}_3].$$

[4]It is called *Grassmannian section* in [**70**].

REMARK 4.3.12 ([**70**]). Since $\operatorname{ran}\mathscr{P}_i = C^\infty(B, \mathcal{W}_i)$, where $\mathcal{W}_i \to B$ has fibre $W_b = \operatorname{ran}P_b$, $\mathscr{P}_2\mathscr{P}_1$ can be seen as an operator $C^\infty(B, \mathcal{W}_1) \to C^\infty(B, \mathcal{W}_2)$.

THEOREM 4.3.13 (Proposition 1, [**52**]). Let $\mathcal{T} \in \Psi^m_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ a family of elliptic operators over B. Then there exist spectral sections for $\mathcal{T}$ if and only if $\operatorname{ind}(\mathcal{T}) = 0$.

REMARK 4.3.14 (§1, [**19**]). $\mathscr{D}_\mathcal{Y}$ in (4.3.1) is elliptic and $\operatorname{ind}(\mathscr{D}_\mathcal{Y}) = 0$ by cobordism invariance, thus there exist spectral sections for $\mathscr{D}_\mathcal{Y}$

DEFINITION 4.3.15 (§3.2, [**69**]). A smooth family of well-posed boundary conditions is a smooth perturbation of the family of Calderón projectors:

$$\mathscr{P} = \mathscr{C} + \mathscr{S} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}'), \qquad \mathscr{S} \in \Psi^{-\infty}_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}').$$

Let $\mathscr{D}_\mathscr{P}$ denote the smooth family of well posed boundary problems. As for the classical case, the existence of the Poisson operator reduces the construction of a vertical parametrix for $\mathscr{D}_\mathscr{P}$ to the construction of a parametrix for the operator $\mathscr{P}\mathscr{C}$ on boundary sections. Therefore:

THEOREM 4.3.16. Let $\mathscr{D}$ be a family of Dirac operators associated to the family of Clifford bundles $\mathcal{E} \to \mathcal{X}$ over B. Let $\mathcal{Y} = \partial\mathcal{X}$ and $\mathscr{P} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ be a family of well-posed boundary conditions. Then:

i) there exists a well-defined virtual bundle $\operatorname{Index}\mathscr{D}_\mathscr{P} \in K^0(B)$ such that $\operatorname{Index}\mathscr{D}_\mathscr{P} = \operatorname{Index}(\mathscr{P}\mathscr{C})$ (Theorem 2.14, [**19**]);
ii) if $\mathscr{P}_1, \mathscr{P}_2 \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ are two well-posed boundary conditions (Theorem 2.13, [**19**]):

$$\operatorname{ind}(\mathscr{D}_1, \mathscr{P}_1) - \operatorname{ind}(\mathscr{D}_1, \mathscr{P}_2) = [\mathscr{P}_2 - \mathscr{P}_1]. \tag{4.3.2}$$

Let $X_i \hookrightarrow \mathcal{X}_i \xrightarrow{\pi_i} B$ be two even-dimensional fibre bundles with common boundary fibre bundle $Y \hookrightarrow \mathcal{Y} \to B$. After choosing a collar neighbourhoods $U_i \hookrightarrow \mathcal{U}_i \xrightarrow{\rho} B$ in $\mathcal{X}_i$ for $\mathcal{Y}$, we can glue them into a fibre bundle $X \hookrightarrow \mathcal{X} \xrightarrow{\rho} B$ with closed fibre $X := X_1 \cup_Y X_2$. As in the single operator case, if $\mathscr{D}_i$ is a family of compatible Dirac operators associated to $\mathcal{X}_i$, we obtain a Dirac operator $\mathscr{D}$ associated to $\mathcal{X}$.

THEOREM 4.3.17 (Theorem 2.10, [**19**]). Let $\mathscr{C}_i := \mathscr{C}_i^+$ be the family of Calderón projectors for $\mathscr{D}_i$, $i = 1, 2$. Then $\operatorname{ind}\mathscr{D} = \operatorname{ind}(\mathscr{C}_2^\perp\mathscr{C}_1) = [\mathscr{C}_1 - \mathscr{C}_2^\perp]$. As a consequence (Theorem 1.1, [**19**]), for $\mathscr{P}_1, \mathscr{P}_2$ two families of generalized spectral sections for $\mathcal{Y}$ as boundary of $\mathcal{X}_1$,

$$\operatorname{ind}\mathscr{D} = \operatorname{ind}\mathscr{D}_{1,\mathscr{P}_1} + \operatorname{ind}\mathscr{D}_{2,\mathscr{P}_2^\perp} + [\mathscr{P}_1 - \mathscr{P}_2] \in K^0(B).$$

REMARK 4.3.18. As for the classical case, (4.3.2) shows that in general the index is not an invariant, as it depends on the spectral section. But like in the classical case, one can use a specific subclass of boundary conditions (if such class exists) in order to remove the dependance on the boundary projection. In fact, as we needed generalised APS boundary conditions in order to split the the index of the signature operator, now we need to use what has been defined in [**42**] as *symmetric spectral section*. It corresponds to an additional assumption which the signature operator hopefully satisfies. As for classical signature, special kind of spectral sections are needed for gluing.

Let now $X_i \hookrightarrow \mathcal{X}_i \overset{\pi_i}{\to} B$ have boundary fibrations $Y_{i-1}^- \sqcup Y_i \hookrightarrow \mathcal{Y}_{i-1}^- \sqcup \mathcal{Y}_i \to B$, $i = 1, 2$. When glued along the common boundary $Y_1 \hookrightarrow \mathcal{Y}_i \to B$, the resulting fibre bundle $X \hookrightarrow \mathcal{X} \to B$, $X = X_1 \cup X_2$ has a non-empty boundary fibre bundle $Y_0 \sqcup Y_2 \hookrightarrow \mathcal{Y}_0 \sqcup \mathcal{Y}_2 \to B$. By Lemma 2.4.12, we can consider diagonal vertical Calderón operators and spectral sections $\mathscr{P} = \{\mathcal{P}_b\}_{b\in B}$ of the form:

$$\mathscr{P} = \mathscr{P}_{0,0}^{\perp} \oplus \mathscr{P}_{1,1} = \begin{pmatrix} \mathscr{P}_{0,0}^{\perp} & 0 \\ 0 & \mathscr{P}_{1,1} \end{pmatrix}.$$

PROPOSITION 4.3.19 (Additivity of the index class). In general:

$$\mathrm{ind}\mathscr{D}_{\mathscr{P}} = \mathrm{ind}\mathscr{D}_{1,\mathscr{P}_1} + \mathrm{ind}\mathscr{D}_{2,\mathscr{P}_2} + [\mathscr{P}_{1,1} - \widetilde{\mathscr{P}}_{1,1}] \in K^0(B). \tag{4.3.3}$$

PROOF.

$$\begin{aligned} \mathrm{ind}\mathscr{D}_{\mathscr{P}} &= [\mathscr{C} - \mathscr{P}] = [\mathscr{C}_{0,0}^{\perp} \oplus \mathscr{C}_{2,2} - \mathscr{P}_{0,0}^{\perp} \oplus \mathscr{P}_{2,2}] \\ &= [(\mathscr{C}_{0,0}^{\perp} - \mathscr{P}_{0,0}^{\perp}) \oplus (\mathscr{C}_{2,2} - \mathscr{P}_{2,2})] = -[\mathscr{C}_{0,0} - \mathscr{P}_{0,0}] + [\mathscr{C}_{2,2} - \mathscr{P}_{2,2}]. \end{aligned}$$

Analogously:

$$\mathrm{ind}\mathscr{D}_{1,\mathscr{P}_1} = -[\mathscr{C}_{0,0} - \mathscr{P}_{0,0}] + [\mathscr{C}_{1,1} - \mathscr{P}_{1,1}],\ \mathrm{ind}\mathscr{D}_{2,\mathscr{P}_2} = -[\mathscr{C}_{1,1} - \widetilde{\mathscr{P}}_{1,1}] + [\mathscr{C}_{2,2} - \mathscr{P}_{2,2}].$$

Hence,

$$\begin{aligned} \mathrm{ind}(\mathscr{D}_1, \mathscr{P}_1) + \mathrm{ind}(\mathscr{D}_2, \mathscr{P}_2) - \mathrm{ind}(\mathscr{D}, \mathscr{P}) &= [\mathscr{C}_{1,1} - \mathscr{P}_{1,1}] - [\mathscr{C}_{1,1} - \widetilde{\mathscr{P}}_{1,1}] \\ &= [\mathscr{C}_{1,1} - \mathscr{P}_{1,1}] + [\widetilde{\mathscr{P}}_{1,1} - \mathscr{C}_{1,1}] \\ &= [\widetilde{\mathscr{P}}_{1,1} - \mathscr{P}_{1,1}] = \mathrm{ind}(\mathscr{P}_{1,1}\widetilde{\mathscr{P}}_{1,1}). \end{aligned}$$

□

4.4. The signature of a fibre bundle as a LogFQFT

Let us consider the strict functor $F^{-\infty}_{\text{vert}} : \mathbf{FCob}_n(B)^* \to \mathbb{C}\text{-}\mathbf{Alg}$ defined as:

$$F^{-\infty}_{\text{vert}}(\mathcal{Y}) := \Psi^{-\infty}_{\text{vert}}(\mathcal{Y}, \mathcal{E}),$$

where $\mathcal{E} \to \mathcal{Y}$ is a family of vector bundles. As in Example 1.4.38, we can define as insertion maps the algebra morphisms $\eta_{\sqcup\mathcal{Z}}\mathcal{Y} : F^{-\infty}_{\text{vert}}(\mathcal{Y}) \hookrightarrow F^{-\infty}_{\text{vert}}(\mathcal{Y} \sqcup \mathcal{Z})$ for $\mathcal{Z} \in \text{Obj}(\mathbf{FCob}_n(B))$ as $\eta_{\sqcup\mathcal{Z}}\mathcal{Y}(\mathcal{T}) = j^*_{\mathcal{Z}} \circ \mathcal{T} \circ i^*_{\mathcal{Z}}$, where $j_{\mathcal{Z}} : \mathcal{Y} \sqcup \mathcal{Z} \to \mathcal{Y}$ is the projection and $i_{\mathcal{Z}} : \mathcal{Y} \hookrightarrow \mathcal{Y} \sqcup \mathcal{Z}$ the inclusion. Consider the morphisms $\widetilde{\eta}_{\sqcup\mathcal{Z}}\mathcal{Y} := K_0(\eta_{\sqcup\mathcal{Z}}\mathcal{Y}) : K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y})) \to K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y} \sqcup \mathcal{Z}))$ induced by K_0 by functoriality. Then, by Lemma 3.2.7, $F^{-\infty}_{\text{vert}}$ is a non-injective higher pretracial monoidal product representation and $(K_0(F^{-\infty}_{\text{vert}}(\mathbf{FCob}_n(B)^*)), \widetilde{\eta}^k_{\sqcup\cdot})$ is a presimplicial set.

Lemma 4.4.1. $\Psi^{-\infty}_{\text{vert}}(\mathcal{Y}, \mathcal{E})$ and $C^\infty(B)$ are Morita equivalent.

Proof. This generalizes the fact that $\Psi^{-\infty}(Y, E)$ and $C^\infty(\{\text{point}\}) = \mathbb{C}$ are Morita equivalent (Example 3.2.16). In fact, $\Psi^{-\infty}_{\text{vert}}(\mathcal{Y}, \mathcal{E})$ is H-unital and, by Schwarz's Kernel Theorem, is naturally identified with $C^\infty(\mathcal{Y} \times \mathcal{Y}, \mathcal{E} \boxtimes \mathcal{E}^*)$, which is a smooth family of complex matrices parametrized by B. Hence, $C^\infty(\mathcal{Y} \times \mathcal{Y}, \mathcal{E} \boxtimes \mathcal{E}^*)$ is Morita equivalent to $\text{End}(C^\infty(B)^N)$, which is Morita equivalent to $C^\infty(B)$.

$\square$

Corollary 4.4.2. $K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y})) \cong K_0(C^\infty(B)) \cong K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y} \sqcup \mathcal{Z}))$ by canonical isomorphisms. In particular, $K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y})) \cong K^0(B)$ and $\widetilde{\eta}^k_{\sqcup}$ are isomorphisms.

Moreover, a fibre bundle isomorphism $\phi : \mathcal{Y} \to \mathcal{Z}$ induces a canonical continuous isomorphism of algebras $\phi_\sharp : F^{-\infty}_{\text{vert}}(\mathcal{Y}) \to F^{-\infty}_{\text{vert}}(\mathcal{Z})$ and pushes-down to a canonical linear isomorphism $\widetilde{\phi_\sharp} : K_0(F^{-\infty}_{\text{vert}}(\mathcal{Y})) \to K_0(F^{-\infty}_{\text{vert}}(\mathcal{Z}))$, hence independent of the initial ϕ.

Proof. If ϕ is a fibre bundle isomorphism, it induces a bundle isomorphism and continuous linear pull-back isomorphism between the corresponding spaces of sections, which provides an isomorphism $\phi_\sharp : F^{-\infty}_{\text{vert}}(\mathcal{Y}) \to F^{-\infty}_{\text{vert}}(\mathcal{Z})$. The rest follows by Lemma 4.4.1.

$\square$

Remark 4.4.3. $F^{-\infty}_{\text{vert}}$ is unoriented. In fact, as $\Psi^{-\infty}(M, E)$ is unoriented (see Lemma 1.4.39), so is $F^{-\infty}_{\text{vert}}(\mathcal{Y})$. It is also tracial with the Chern character as a trace.

Consider a representative $\mathcal{X} \to B$, with $2m$-dimensional fibre X, of a fibred cobordism class in $\text{mor}_{\mathbf{FCob}_n(B)}(\mathcal{M}_0, \mathcal{M}_1)$. As usual, we consider a fibred collar near its boundary $Y \hookrightarrow \mathcal{Y} \to B \in \mathrm{obj}(\mathbf{FCob}_n(B))$, with vertical product structure, i.e. for the vertical metric $g^{T(\mathcal{X}/B)}$. Recall that after choosing a connection we can decompose $T\mathcal{X} \cong T(\mathcal{X}/B) \oplus T^H\mathcal{X}$ (Remark 4.1.8); this yields the decomposition

$$\Lambda(\mathcal{X}) := \Lambda(T^*\mathcal{X}) \cong \Lambda_\pi(\mathcal{X}) \otimes \pi^*\Lambda(B), \quad \text{where} \quad \Lambda_\pi(\mathcal{X}) := \Lambda(T^*(\mathcal{X}/B)).$$

Let $\mathcal{E} \to \mathcal{X}$ be a smooth family of vector bundles which is *flat* along the fibres. By Remark 4.1.7, the smooth sections of $\Lambda_\pi(\mathcal{X}) \otimes \mathcal{E} \to \mathcal{X}$ correspond to the smooth sections of $\mathcal{W} := \pi_*(\Lambda_\pi(\mathcal{X}) \otimes \mathcal{E}) \to B$, i.e.

$$\Omega_{\mathrm{vert}}(\mathcal{X}, \mathcal{E}) := C^\infty(\mathcal{X}, \Lambda_\pi(\mathcal{X}) \otimes \mathcal{E}) = C^\infty(B, \mathcal{W}).$$

Recall that the fibre of $\mathcal{W} \to B$ is $\Omega(X, E)$. Then $\Omega_{\mathrm{vert}}(\mathcal{X}, \mathcal{E})$ is a subspace of the total space of smooth forms $\Omega(\mathcal{X}, \mathcal{E}) := C^\infty(\mathcal{X}, \Lambda(\mathcal{X}) \otimes \mathcal{E})$, corresponding to sections that vanish under interior multiplication with horizontal vectors (§3, [**9**]).

Let $d^X := (d_b)_{b\in B}$ be the associated smooth family of exterior derivatives. Since we assume a vertical Riemannian metric, we obtain a smooth family of Hodge operators $*^M := (*_b)_{b\in B}$ and an associated family of coderivatives $\delta^X := (\delta)_{b\in B}$ in the obvious way (see §3, [**9**], for a detailed description), thus obtaining the family of Dirac operators $\mathscr{D} := (d_b + \delta_b)_{b\in B} \in \Psi^1_{\mathrm{vert}}(\mathcal{X}, \Lambda_\pi(\mathcal{X}) \otimes \mathcal{E})$ acting on the vertical smooth differential forms.

Definition 4.4.4. The operator $\mathscr{D}^{\mathrm{Sign}}$, defined as the restriction of the family $\mathscr{D}$ to $\Lambda^+_\pi(\mathcal{X}) \otimes \mathcal{E}$ of the $\mathbb{Z}_2$-grading induced by the fibrewise Hodge operator, $\Lambda_\pi(\mathcal{X}) = \Lambda^+_\pi(\mathcal{X}) \oplus \Lambda^-_\pi(\mathcal{X})$, is called *(twisted) family signature operator.*

Let us consider the restriction $\mathscr{D}^{\mathrm{Sign}}_{\mathcal{Y}}$, which is a twisted (odd) family signature over the boundary. By cobordism invariance, $\mathrm{ind}\mathscr{D}^{\mathrm{Sign}}_{\mathcal{Y}} = 0$ and we have a non-empty grassmannian of spectral sections (Remark 4.3.14).

Let $(\mathscr{D}^{\mathrm{Sign}}_{\mathcal{Y}})^2 := (\Delta^{\mathrm{sign}}_{Y_b})_{b\in B}$ be the (twisted) family of signature Laplacians. Since we assumed that the fibre X is $2m$-dimensional, the boundary $Y = \partial X$ has dimension $2m-1$.

Proposition 4.4.5 (Propositions 1.2 & 1.3, [**42**]). If $\ker(\Delta^{\mathrm{sign}}_{m,Y_b})$, i.e. the space of harmonic forms in degree m, has constant dimension with respect to $b \in B$, then there exist spectral sections, called *symmetric*[5], such that for any two such sections $\mathscr{P}, \mathscr{Q} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \Lambda_\pi(\mathcal{Y}) \otimes \mathcal{E}')$,

$$[\mathscr{P} - \mathscr{Q}] = 0 \quad \text{in} \quad K^0(B) \otimes \mathbb{Q}.$$

[5]For a detailed exposition and explanation of the name, see [**42**].

Clearly, if $K^0(B)$ is torsion-free, then $[\mathscr{P}-\mathscr{Q}]=0$ in $K^0(B)$.

REMARK 4.4.6 ([**43**]). Symmetric spectral sections generalize the idea of generalized APS projections (which are subordinated to a Lagrangian subspace for the signature operator on the boundary), which allowed an additivity formula. Importantly, they provide a homotopy invariant index and Chern character, and the homotopy invariance of the scalar signatures that may arise from them, such as the signature of the total space of a fibre bundle (see Remark 4.4.12). This is related to higher (Novikov) signatures, and therefore we will say more in the next Chapter. The results there will be in fact analogous to the family case.

Let $\mathscr{C}:=\mathscr{C}^+\in\Psi^0_{\mathrm{vert}}(\mathcal{Y},\Lambda^+_\pi(\mathcal{Y})\otimes\mathcal{E}')$ denote the family of Calderón projectors associated to the family signature operator and $\mathscr{P}$ a symmetric spectral section. We define a universal LogTQFT:

$$u\text{-}\log^{\mathrm{Sign}}:\mathcal{N}\mathbf{FCob}_n(B)\to K_0(F^{-\infty}_{\mathrm{vert}}(\mathbf{FCob}^*_n(B)))\otimes\mathbb{Q}$$

by setting $u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0\sqcup\mathcal{M}_1}:\mathrm{mor}(\mathcal{M}_0,\mathcal{M}_1)\to K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0\sqcup\mathcal{M}_1))\otimes\mathbb{Q}$ as

$$u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0\sqcup\mathcal{M}_1}\mathcal{X}:=\widetilde{\phi}_{\sharp,\mathcal{M}_0\sqcup\mathcal{M}_1}([\mathscr{C}-\mathscr{P}])\in K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0\sqcup\mathcal{M}_1))\otimes\mathbb{Q}, \tag{4.4.1}$$

with $\widetilde{\phi}_{\sharp,\mathcal{M}_0\sqcup\mathcal{M}_1}$ the canonical isomorphism $K_0(F^{-\infty}_{\mathrm{vert}}(\partial\mathcal{X}))\cong K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0\sqcup\mathcal{M}_1))$.

THEOREM 4.4.7. (4.4.1) defines a universal LogTQFT, i.e. with respect to gluing along a common boundary, in $K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0\sqcup\mathcal{M}_1\sqcup\mathcal{M}_2))\otimes\mathbb{Q}$ we have:

$$\widetilde{\eta}_{\mathcal{M}_1}u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0\sqcup\mathcal{M}_2}\mathcal{X}=\widetilde{\eta}_{\mathcal{M}_2}u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_0\sqcup\mathcal{M}_1}\mathcal{X}_1+\widetilde{\eta}_{\mathcal{M}_0}u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}_1\sqcup\mathcal{M}_2}\mathcal{X}_2.$$

PROOF. The $\widetilde{\eta}_{\mathcal{M}_i}$ are isomorphisms into $K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M}_0\sqcup\mathcal{M}_1\sqcup\mathcal{M}_2))\cong K^0(B)$, where we have:

$$[\mathscr{C}-\mathscr{P}]=[\mathscr{C}_1-\mathscr{P}_1]+[\mathscr{C}_2-\mathscr{P}_2]\in K^0(B)\otimes\mathbb{Q}.$$

from (4.3.3) and Proposition 4.4.5.

□

REMARK 4.4.8. If $\mathcal{X}$ is closed, $\mathcal{Y}$ codimension 1 closed sub-bundle such that $\mathcal{Y}\cong\mathcal{M}$, then by Theorem 4.3.17:

$$u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}}\mathcal{X}:=\widetilde{\phi}_{\sharp,\mathcal{M}}([\mathscr{C}_1-\mathscr{C}_2^{\perp}])\in K_0(F^{-\infty}_{\mathrm{vert}}(\mathcal{M})).$$

Since $F^{-\infty}_{\mathrm{vert}}(\mathcal{M})$ is equivalent to the commutative C^*-algebra $C^\infty(M)$, the Chern character of §3.2.2 corresponds to the classical $\mathrm{ch}_*:K^0(B)\to H^{2*}(B)$, defined via a superconnection (Remark 3.2.10).

DEFINITION 4.4.9 (Definition 1.37, [**8**], and §1.3, [**70**]). A superconnection on $\mathcal{W}$ adapted to $\mathscr{P} \in \mathcal{A}^0(B, \mathrm{End}(\mathcal{W}))$ is an odd-parity first order differential operator $\mathbb{A}$ on the graded complex $\mathcal{A}(B, \mathcal{W})$ such that $\mathbb{A}(\omega \wedge s) = d\omega \wedge s + (-1)^{|\omega|}\omega \wedge \mathbb{A}(s)$ for $\omega \in \Omega(B)$ and $s \in \mathcal{A}(B, \mathcal{W})$, and with $\mathbb{A}_{[0]} = \mathscr{P}$, where $\mathbb{A}_{[i]}$ is the component of $\mathbb{A}$ which raises form degree by i. The curvature of $\mathbb{A}$ is the even-parity element $\mathbb{A}^2$ of $\mathcal{A}(B, \mathrm{End}(\mathcal{W}))$.

THEOREM 4.4.10 (Theorem 1.4, [**70**]). Let $\mathscr{D} \in \Psi^1_{\mathrm{vert}}(\mathcal{X}, \mathcal{E})$ be elliptic and $\mathscr{P} \in \Psi^0_{\mathrm{vert}}(\mathcal{Y}, \mathcal{E}')$ be a spectral section. Then:

$$\mathrm{ch}(\mathrm{ind}\mathscr{D}_{\mathscr{P}}) = \mathrm{ch}([\mathscr{C} - \mathscr{P}]) = \sum_{k=0}^{\dim B} \frac{1}{k!}\mathrm{Tr}_{\mathcal{Y}/B}\left(R^k_{\mathscr{C}} - R^k_{\mathscr{P}}\right) \in H^{2*}(B),$$

where $R_{\mathscr{P}} := \left(\mathscr{P} \cdot \nabla^{\mathcal{W}} \cdot \mathscr{P}\right)^2 \in \mathcal{A}^2(B, \Psi^0(\mathcal{Y}, \mathcal{E}'))$ is the curvature of the sub-bundle $\mathrm{ran}(\mathscr{P})$.

REMARK 4.4.11. The theorem uses canonically defined superconnections $\mathbb{A}_{\mathscr{C}}$ and $\mathbb{A}_{\mathscr{P}}$. In particular, $R^0_{\mathscr{P}} = \mathscr{P}$ and

$$\mathrm{ch}(\mathrm{ind}\mathscr{D}_{\mathscr{P}})_{[0]} = \mathrm{Tr}_{\mathcal{Y}/B}\left(\mathscr{C} - \mathscr{P}\right) \in H^0(B)$$

is constant over B and corresponds to the pointwise index $\mathrm{ind}(\mathcal{C}_b - \mathcal{P}_b) \in \mathbb{Z}$, which is $\sigma(X)$, the signature of the fibre.

REMARK 4.4.12. Therefore, the signature of a fibre bundle $\mathcal{X} \to B$ can arise as a log-determinant of the higher LogTQFT $\mathrm{ch}(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}} \mathcal{X}) \in H^{2*}(B)$. In fact, let $L(B) \in H^*(B)$ denotes the Hirzebruch L-class of B and consider the Poincaré dual of $\mathrm{ch}(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}} \mathcal{X})$, i.e. $\mathrm{ch}(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}} \mathcal{X}) \cap [B] \in H_*(B)$, where $[B] \in H_{\dim B}(B)$ is the fundamental class of B. Then, by Kronecker pairing:

$$\begin{aligned}\langle L(B), \mathrm{ch}(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}} \mathcal{X}) \cap [B]\rangle &= \langle \mathrm{ch}(u\text{-}\log^{\mathrm{Sign}}_{\mathcal{M}} \mathcal{X}) \wedge L(B), [B]\rangle \\ &= \langle L(T_\pi \mathcal{X}) \wedge \pi^* L(B), [\mathcal{X}]\rangle = \sigma(\mathcal{X}).\end{aligned}$$

It is a oriented homotopy invariant of the fibre bundle $\mathcal{X}$, as so is the right-hand side of (4.4.1), by results in [**43**] (Remark 4.4.6).

REMARK 4.4.13. These approach can be used for the family de Rham operator with relative boundary conditions, thus generalizing the result of Chapter 2. However, the cohomology bundle is flat, hence all classes of $\mathrm{ch}(\mathrm{ind}\mathscr{D}^{\mathrm{dR}}_{\mathcal{R}})$ vanish, except for that of order zero, which corresponds to the Euler characteristic of the fibre.

CHAPTER 5

Other Higher LogTQFT

As for fibre bundles, one can define a log-functor for *singular manifolds*, i.e. continuous maps $M \to B$ from a manifold M to a path connected space B. In particular, we will consider the case that $M \to B$ is a Galois covering. This moves the problem to the setting of non-commutative geometry and our attempt here is to see higher Novikov signatures as log-characters of a higher LogTQFT.

5.1. Galois Γ-coverings and LogHQFTs

DEFINITION 5.1.1. Let M be a manifold. A covering $\widetilde{M} \to M$ is called *Galois* (or *regular* or *normal*) if there exists a discrete and finitely presented group Γ acting freely and transitively on the fibres. In particular, it is a principal Γ-bundle.

EXAMPLE 5.1.2. The universal cover is a Galois covering, where $\Gamma = \pi_1(M)$.

REMARK 5.1.3. By the *Classifying Theorem for Principal Bundles* (Appendix B, [**40**]), isomorphism classes of Galois covering are bijectively associated to homotopy classes of *classifying maps* $r : M \to B\Gamma$, i.e. continuous maps with values in the classifying space[1] of Γ, which is uniquely defined modulo homotopy. Therefore, we will identify Galois coverings with the pair (M, r), which is the notation for a singular manifold[2] ([**16**]).

DEFINITION 5.1.4 (Definition 5.1, [**45**]). Let (M, r) and (M', s) be closed oriented Γ-coverings. They are *oriented homotopy equivalent* if there exists a oriented homotopy equivalence $h : M \to M'$ such that $s \circ h \simeq r$, i.e. $s \circ h$ and r are homotopic.

DEFINITION 5.1.5 (Definition 5.2, [**45**]). Let $\partial M, \partial M' \neq \emptyset$ and such that there exist orientation preserving diffeomorphisms $\phi, \psi : \partial M \to \partial M'$. Then two

[1] A *classifying space* for a group Γ is a connected topological space $B\Gamma$ together with a principal Γ-bundle $E\Gamma \to B\Gamma$ such tha for any compact Hausdorff space X there is a bijective correspondence between the equivalence classes of principal Γ-bundles over X and the homotopy classes of maps $X \to B\Gamma$ (Definition B.I, [**40**]).

[2] Or a $B\Gamma$-manifold, as they are called in [**66**].

Γ-coverings $r : M \cup_\phi M' \to B\Gamma$ and $s : M \cup_\psi M' \to B\Gamma$ are said to be *cut-and-paste equivalent* if $r_{|M} \simeq s_{|M}$ and $r_{|M'} \simeq s_{|M'}$.

The above definition can be extended to $M \cup_\phi M'$ when $\partial(M \cup_\phi M') \neq \emptyset$ for homotopies relative to the boundary.

REMARK 5.1.6. Clearly two closed oriented Galois Γ-coverings (M, r) and (M', s) are diffeomorphic if there exists an orientation preserving diffeomorphism $\psi : M \to M'$ such that $r = s \circ \psi$. In particular, a diffeomorphism must fix the boundaries, i.e. $\psi_{|\partial M} : \partial M \overset{\cong}{\to} \partial M'$. Moreover, we can define the disjoint union of Γ-coverings (over manifolds with or without boundary) and the covering with inverse orientation as in Definition 4.1.9. Finally, (M, r), $(M \sqcup \emptyset, r)$ and $(\emptyset \sqcup M, r)$ are naturally diffeomorphic (Remark 4.1.10).

DEFINITION 5.1.7. An oriented Galois Γ-covering (M, r) *bords* (or *is a boundary*) if there exist an oriented manifold W such that $M \overset{\psi}{\cong} \partial W$, and a homotopy class of continuous maps $R : W \to B\Gamma$ relative to the boundary such that $R_{|\partial W} \circ \psi = r$. Therefore, two oriented Galois Γ-coverings (M_1, r_1) and (M_2, r_2) are *bordant* if and only if $(M_1^- \sqcup M_2, r_1 \sqcup r_2)$ bords (W, R), which is called *BΓ-cobordism*, following [**66**].

Let $\mathbf{Diff}(B\Gamma)$ be the category of oriented Galois Γ-coverings and diffeomorphisms between them. When endowed with disjoint union of Definition 4.1.9, it becomes a symmetric monoidal category and a subcategory of **Diff**.

DEFINITION 5.1.8. Consider two $B\Gamma$-cobordisms $(W, F) : (M_1, f_1) \to (M_2, f_2)$ and $(W', F') : (M_2', f_2') \to (M_3, f_3)$ with diffeomorphic boundary components $(M_2, f_2) \overset{\psi}{\cong} (M_2', f_2')$. Then their composition is the $B\Gamma$-cobordism $(W \cup_\psi W', G)$ such that:

$$G(w) := F \cdot F'(w) := \begin{cases} F(w) & \text{if } w \in W \\ F'(w) & \text{if } w \in W'. \end{cases}$$

If (M, f) is a closed oriented Galois Γ-covering, then the identity for the composition is the $B\Gamma$-cobordism:

$$([0,1] \times M, 1_f) : (M, f) \to (M, f) \text{ with } 1_f(t, m) = f(m).$$

DEFINITION 5.1.9 (§1, [**66**]). $(n-1)$-dimensional oriented Galois Γ-coverings and $B\Gamma$-cobordisms define the category (enriched over categories) $\mathbf{HCob}_n(B\Gamma)$ of *homotopy n-cobordism* over $B\Gamma$. Together with disjoint union, it is a symmetric monoidal category whose objects are oriented Galois Γ-coveringof dimension $n-1$, and whose morphisms are (compositions of) $B\Gamma$-cobordisms and oriented Galois Γ-covering diffeomorphisms.

For a precise construction of $\mathbf{HCob}_n(X)$, X a path connected space, we refer to the Appendix of [**66**]. We remark that it is constructed as a category enriched over categories (as $\mathbf{FCob}_n(X)$), out of the category $\mathbf{Diff}(X)$. In particular, $\mathbf{HCob}_n(X) \subseteq \mathbf{Cob}_n$. Thus, once we choose a good monoidal product representation F, we can define a higher log-functor. In particular:

DEFINITION 5.1.10. A log-functor $\log : \mathcal{N}\mathbf{HCob}_m(X) \to HC_n\left(F(\mathbf{HCob}^*_m(X))\right)$ is called *Logarithmic Homotopy Quantum Field Theory* (LogHQFT) of dimension m and order n.

As for TQFTs, LogHQFTs can define HQFTs, at least in a weak sense.

LEMMA 5.1.11. Let $F : \mathbf{HCob}^*_m(X) \to \mathbf{Ring}_{\mathbf{Add}}$ be a pretracial monoidal product representation and $\log : \mathcal{N}\mathbf{HCob}_m(X) \to \left(HC_n\left(F(\mathbf{HCob}^*_n(X))\right), \tau\right)$ an associated LogHQFT. If $\epsilon : \mathrm{end}(1_{\mathbf{A}}) \to R$ is an exponential map into a commutative ring, then there exists a symmetric monoidal functor $Z_{\log,\tau,\epsilon} : \mathbf{HCob}_n(X) \to R\text{-}\mathbf{Mod}$, i.e. a HQFT defined as:

$$Z_{\log,\tau,\epsilon}(M,f) = R \qquad Z(\psi) = R^* \qquad Z_{\log,\tau,\epsilon}(W,F) = \epsilon(\tau(\log(W,F))).$$

PROOF. This follows directly from the definition and the log-additivity, as for Lemma 1.4.37.

□

5.2. Dirac operators associated to Galois coverings

DEFINITION 5.2.1 (§7.1, [**45**]). Let $B(\ell^2(\Gamma))$ be the algebra of bounded operators of $\ell^2(\Gamma)$ and let $\mathbb{C}\Gamma$ be the group ring of Γ. Then its completion in $B(\ell^2(\Gamma))$ is a unital C^*-algebra called *reduced group C^*-algebra $C^*_r\Gamma$*.

For M closed, let (M,r) be a Galois covering and $\eth : C^\infty(M,E) \to C^\infty(M,E)$ be the Dirac operator associated to a Clifford module $E \to M$, with unitary connection ∇^E. Since $r : M \to B\Gamma$ corresponds to a Γ-covering $\widetilde{M} \xrightarrow{\rho} M$ (Remark 5.1.3), $\eth$ can be lifted to a Γ-invariant operator $\widetilde{\eth} : C^\infty(\widetilde{M},\widetilde{E}) \to C^\infty(\widetilde{M},\widetilde{E})$, with $\widetilde{E} := \rho^* E$ a Γ-equivariant bundle. Moreover, since Γ acts on the right on $C^*_r\Gamma$ by translation and on the left on $\widetilde{M}$ by deck transformation, we have an associated bundle of finitely generated projective $C^*_r\Gamma$-modules (check):

$$\mathcal{V} := C^*_r\Gamma \times_\Gamma \widetilde{M} \to M, \quad \text{with fibre } C^*_r\Gamma.$$

Let us consider smooth sections $s \in C^\infty(M, E \otimes \mathcal{V})$. Therefore, if $h(\cdot,\cdot)$ denotes the Hermitian metric on E, there is a $C^*_r\Gamma$-valued inner product, which is defined on an open neighbourhood U by:

$$\langle s_1, s_2 \rangle := \int_U h(s_1, s_2) \in C^*_r\Gamma, \qquad s_1, s_2 \in C^\infty(U, (E \otimes \mathcal{V})_{|U}).$$

Hence $C^\infty(M, E\otimes\mathcal{V})$ is a left $C^*_r\Gamma$-module, and with such inner product it becomes a pre-Hilbert $C^*_r\Gamma$-module:

Definition 5.2.2 (15.1.1 & 15.1.5, [**85**]). Let $\mathcal{B}$ be a C^* algebra. A *pre-Hilbert $\mathcal{B}$-module* is a right $\mathcal{B}$-module $\mathcal{H}$ with a compatible $\mathbb{C}$-vector space structure, together with a $\mathcal{B}$-inner product $\mathcal{H} \times \mathcal{H} \to \mathcal{B}$, i.e a sesquilinear positive define form that respects the module action. A *Hilbert $\mathcal{B}$-module* is a pre-Hilbert module that is complete with respect to the norm $\|x\| := \sqrt{\|\langle x, x\rangle\|}$.

The Hilbert module completion of $C^\infty(M, E \otimes \mathcal{V})$ is denoted $L^2_{C^*_r\Gamma}(M, E \otimes \mathcal{V})$.

Remark 5.2.3 (§7.3, [**45**]). As $C^*_r\Gamma \times \widetilde{M} \to \widetilde{M}$ has trivial flat connection, $\mathcal{V} \to M$ has a (non-trivial) flat connection $\nabla^{\mathcal{V}}$. Hence $E \otimes \mathcal{V} \to M$ has connection $\nabla^E \otimes I + I \otimes \nabla^{\mathcal{V}}$. This defines a $C^*_r\Gamma$-linear Dirac operator ((7.1) in [**45**]):

$$\mathscr{D}_{(M,r)} : C^\infty(M, E \otimes \mathcal{V}) \to C^\infty(M, E \otimes \mathcal{V}).$$

Remark 5.2.4 (§7.3, [**45**]). Since $C^\infty(M, E \otimes \mathcal{V})$ can also be completed into Sobolev $C^*_r\Gamma$-modules $H^s_{C^*_r\Gamma}(M, E \otimes \mathcal{V})$, $\mathscr{D}_{(M,r)}$ extends to a bounded $C^*_r\Gamma$-linear operator:

$$\mathscr{D}_{(M,r)} : H^1_{C^*_r\Gamma}(M, E \otimes \mathcal{V}) \to L^2_{C^*_r\Gamma}(M, E \otimes \mathcal{V}).$$

In particular, if E is $\mathbb{Z}_2$-graded:

$$\mathscr{D}_{(M,r)} = \begin{pmatrix} 0 & \mathscr{D}^-_{(M,r)} \\ \mathscr{D}^+_{(M,r)} & 0 \end{pmatrix},$$

with $\mathscr{D}^\pm_{(M,r)} : C^\infty(M, E^\pm \otimes \mathcal{V}) \to C^\infty(M, E^\mp \otimes \mathcal{V})$ $C^*_r\Gamma$-linear.

Example 5.2.5. The signature operator $\eth^{\mathrm{Sign}} : \Omega^+(M) \to \Omega^-(M)$ defines a *twisted signature operator* $\mathscr{D}^{\mathrm{Sign}}_{(M,r)}$ on the twisted signature bundle $\Lambda^+(M)\otimes\mathcal{V} \to M$.

For E, F vector bundles over M, set $\mathcal{E} := E \otimes \mathcal{V}$ and $\mathcal{F} := F \otimes \mathcal{V}$. Then there is a well-developed *Mishchenko-Fomenko pseudodifferential calculus*, [**54**]. The algebra of ψdodifferential $C_r^*\Gamma$-linear opertors $\Psi^*_{C_r^*\Gamma}(M; \mathcal{E}, \mathcal{F})$ contains the subalgebra of elliptic $C_r^*\Gamma$-linear differential opertors, which we denote $\mathrm{Diff}^*_{C_r^*\Gamma}(M; \mathcal{E}, \mathcal{F})$, following [**45**]. Hence, $\mathscr{D}^{\mathrm{sign}}_{(M,r)}$ belongs to $\mathrm{Diff}^1_{C_r^*\Gamma}(M; \Lambda(M)^+ \otimes \mathcal{V}, \Lambda(M)^- \otimes \mathcal{V})$. In particular, there exist parametrices for the operators in $\mathrm{Diff}^*_{C_r^*\Gamma}(M; \mathcal{E}, \mathcal{F})$.

Finally, in this case as well there are decomposition formulae:

$$C^\infty(M, \mathcal{E}) = \mathcal{I}_+ \oplus \mathcal{I}_+^\perp \text{ and } C^\infty(M, \mathcal{F}) = \mathcal{I}_- + \mathscr{D}_{(M,r)}(\mathcal{I}_+^\perp),$$

with $\mathcal{I}_\pm$ finitely generated projective $C_r^*\Gamma$-modules. Note that the second decomposition is not necessarily orthogonal, but $\mathscr{D}_{(M,r)}$ induces an isomorphisms between $\mathcal{I}_+^\perp$ and $\mathscr{D}_{(M,r)}(\mathcal{I}_+^\perp)$.

DEFINITION 5.2.6. The *index class* of $\mathscr{D}_{(M,r)}$ *à la Mishchenko-Fomenko* is:

$$\mathrm{ind}(\mathscr{D}_{(M,r)}) = [\mathcal{I}_+] - [\mathcal{I}_-] \in K_0(C_r^*\Gamma).$$

REMARK 5.2.7. Let $\mathscr{P}_+$ be the orthogonal projection onto $\mathcal{I}_+$ and $\mathscr{P}_-$ be the projection onto $\mathcal{I}_-$ along $\mathscr{D}_{(M,r)}(\mathcal{I}_+^\perp)$. Then $\mathscr{P}_\pm$ are smoothing pseudodifferential operators of the Mishchenko-Fomenko pseudodifferential calculus and hence define a smoothing perturbation $\mathscr{R} = -\mathscr{P}_- \mathscr{D}_{(M,r)} \mathscr{P}_+$ of $\mathscr{D}_{(M,r)}$.

Therefore, since $\ker(\mathscr{D}_{(M,r)})$ and $\mathrm{coker}(\mathscr{D}_{(M,r)})$ are not necessarily finitely generated projective modules,

$$\mathrm{ind}(\mathscr{D}_{(M,r)}) = [\ker(\mathscr{D}_{(M,r)} + \mathcal{R})] - [\mathrm{coker}(\mathscr{D}_{(M,r)} + \mathcal{R})] \in K_0(C_r^*\Gamma),$$

independently of the perturbation $\mathscr{R}$.

Let now (M, r) be $2m$-dimensional with non-empty boundary ∂M and a product type close to it. As for the closed case, given a Clifford bundle we can define a twisted Dirac operator $\mathscr{D}_{(M,r)}$. Let $\mathscr{D}_{(\partial M, r_\partial)} : C^\infty(\partial M, \mathcal{E}') \to C^\infty(M, \mathcal{E}')$ be the associated boundary Dirac operator, where $r_\partial := r_{|\partial M}$ and $\mathcal{E}' := \mathcal{E}_{|\partial M}$, corresponding to the boundary operator for $\mathfrak{d}$.

As for the family case, boundary conditions are realized via spectral sections, which can be defined since we can use functional calculus in this context as well. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ a full Hilbert $\mathcal{A}$-module. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded $\mathcal{A}$-linear adjointable operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the ideal of such operators that are also compact. If $\mathcal{D}$ is a densely defined unbounded self-adjoint

$\mathcal{A}$-linear regular operator on $\mathcal{H}$, then continuous functional calculus on $\mathcal{D}$ is well defined: for any $f \in C(\mathbb{R}, \mathbb{C})$ such that $\exists \lim_{t\to\infty} f(t) < \infty$, $f(\mathcal{D})$ is in $\mathcal{B}(\mathcal{H})$.

DEFINITION 5.2.8 (Definition 2.1, [**41**]). A spectral section for $\mathcal{D}$ is a self-adjoint projection $\mathscr{P} \in \mathcal{B}(\mathcal{H})$ such that there exist smooth spectral cuts[3] χ_1, χ_2 such that $\chi_2(t) = 1$ for $t \in \operatorname{supp}(\chi_1)$ and:

$$\operatorname{im}\chi_1(\mathcal{D}) \subset \operatorname{im}\mathscr{P} \subset \operatorname{im}\chi_2(\mathcal{D}).$$

A criterion for the existence of a spectral cut is the vanishing of the index.

THEOREM 5.2.9 (Theorem 2.2 and Proposition 2.8, [**41**]). There exists one spectral section $\mathscr{P}$ for $\mathcal{D}$, and hence infinitely many, if and only if $\operatorname{ind}(\mathcal{D}) = 0$ in $K_1(\mathcal{A})$.

REMARK 5.2.10. $\mathscr{D}_{(\partial M, r_\partial)}$ is a densely defined unbounded self-adjoint $C^*_r\Gamma$-linear regular operator on $L^2_{C^*_r\Gamma}(\partial M, \mathcal{E}')$ (Proposition 2.3, [**41**]). Moreover, Cobordism Invariance holds also in the context of Galois coverings and thus we have:

$$\operatorname{ind}(\mathscr{D}_{(\partial M, r_\partial)}) = 0 \in K_1(C^*_r\Gamma).$$

Hence, there always exist spectral sections $\mathscr{P} \in \Psi^0_{C^*_r\Gamma}(\partial M; \mathcal{E}', \mathcal{E}')$ for $\mathscr{D}_{(\partial M, r_\partial)}$ (Theorem 2.7 (1), [**41**]).

THEOREM 5.2.11 ([**45**]; 7.6, [**41**]). Let $\mathscr{D}_{(M,r)}$ a twisted Dirac operator associated to a Galois covering (M, r) with non-empty boundary. Let $\mathscr{P} \in \Psi^0_{C^*_r\Gamma}(\partial M; \mathcal{E}', \mathcal{E}')$ be a spectral section for the boundary Dirac operator $\mathscr{D}_{(\partial M, r_\partial)}$. Then $\mathscr{D}_{(M,r)}$ with domain $C^\infty(M, \mathcal{E}; \mathscr{P}) := \{s \in C^\infty(M, \mathcal{E}) |\ \mathscr{P}s_{|\partial M} = 0\}$ has a well defined index $\operatorname{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) \in K_0(C^*_r\Gamma)$, depending only on $\mathscr{P}$.

The classical index formulas hold also in this context.

THEOREM 5.2.12 (Theorem 6, [**44**]). Let $\mathscr{P}, \mathscr{Q} \in \Psi^0_{C^*_r\Gamma}(\partial M; \mathcal{E}', \mathcal{E}')$ be spectral section for $\mathscr{D}_{(\partial M, r_\partial)}$. Then:

$$\operatorname{ind}(\mathscr{D}_{(M,r)}, \mathcal{P}) - \operatorname{ind}(\mathscr{D}_{(M,r)}, \mathcal{Q}) = [\mathcal{Q} - \mathcal{P}] \in K_0(C^*_r\Gamma). \tag{5.2.1}$$

THEOREM 5.2.13 (Theorem 8 & 9, [**44**]). If (M, r) is a Galois covering split into two Galois coverings with boundary (M_i, r_i), where $r_i = r_{|M_i}$ for $i = 1, 2$, by a 1-codimensional manifold N, then:

$$\operatorname{ind}(\mathscr{D}_{(M,r)}) = \operatorname{ind}(\mathscr{D}_{(M_1,r_1)}, \mathscr{P}) + \operatorname{ind}(\mathscr{D}_{(M_2,r_2)}, \mathscr{Q}^\perp) + [\mathscr{P} - \mathscr{Q}].$$

[3]A *smooth spectral cut* is a function $\chi \in C^\infty(\mathbb{R}, [0, 1])$ such that for some real $s_1 < s_2$, $\chi(t) = 0$ if $t \leq s_1$ and $\chi(t) = 1$ if $t \geq s_2$ (§2, [**41**]).

COROLLARY 5.2.14. Let (M_i, r_i), $i = 1, 2$, be Galois Γ-coverings such that $\partial M_i = Y_{i-1} \sqcup Y_i$ and let $(M = M_1 \cup_{Y_1} M_2, R)$, $R = r_1 \cdot r_2$, be their composition. We consider the following spectral sections for M, M_1, and M_2, respectively:

$$\mathscr{P} = \begin{pmatrix} \mathscr{P}_{0,0} & 0 \\ 0 & \mathscr{P}_{2,2} \end{pmatrix}, \quad \mathscr{P}_1 = \begin{pmatrix} \mathscr{P}_{0,0} & 0 \\ 0 & \mathscr{P}_{1,1} \end{pmatrix}, \quad \mathscr{P}_2 = \begin{pmatrix} I - \mathscr{Q}_{1,1} & 0 \\ 0 & \mathscr{P}_{2,2} \end{pmatrix}.$$

Then:

$$\mathrm{ind}(\mathscr{D}_{(M,R)}, \mathscr{P}) = \mathrm{ind}(\mathscr{D}_{(M_1,r_1)}, \mathscr{P}_1) + \mathrm{ind}(\mathscr{D}_{(M_2,r_2)}, \mathscr{P}_2) + [\mathscr{P}_{1,1} - \mathscr{Q}_{1,1}].$$

REMARK 5.2.15. In the classical or family case, the proof can be based on the fact that the index of the realization via a spectral section coincides with the K-theory class of the difference between Calderón operator and the spectral section itself. As a matter of fact, a Calderón projector $\mathscr{C}$ exists also for these elliptic value problems over C^*-algebras, and is obtained essentially from the classical proof (see [**10**]) by methods allowed for Hilber modules over a C^*-algebra $\mathcal{B}$ (see [**1**]). However it is still only conjectured that:

$$\mathrm{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) = [\mathscr{C} - \mathscr{P}]$$

On the other hand, given a spectral section $\mathscr{P}$, there is a well-defined Grassmannian $\mathcal{G}_{\mathscr{P}} := \{\mathscr{Q} \text{ spectral section} \mid \mathscr{Q} - \mathscr{P} \text{ compact}\}$, whose connected components are in bijective correspondance with the classes in $K_0(\mathcal{B})$ via the map $\mathscr{Q} \to [\mathscr{P} - \mathscr{Q}]$ (See [**30**]). Then, by (5.2.1) there exist a spectral section $\widetilde{\mathscr{P}} \in \mathcal{G}_{\mathscr{P}}$ corresponding to the class $\mathrm{ind}(\mathscr{D}_{(M,r)}, \mathscr{P})$, i.e. $[\mathscr{P} - \widetilde{\mathscr{P}}] = -\mathrm{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) \in K_0(\mathcal{B})$, which yields:

$$\mathrm{ind}(\mathscr{D}_{(M,r)}, \widetilde{\mathscr{P}}) = \mathrm{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) + [\mathscr{P} - \widetilde{\mathscr{P}}] = 0.$$

Therefore, $\mathrm{ind}(\mathscr{D}_{(M,r)}, \widetilde{\mathscr{P}})$ can be expressed as a class depending only on the spectral sections, and in particular, the index depends on the boundary and the quasi-additivity of Corollary 5.2.14 follows.

5.2.1. Smoothing of the algebra. From Remark 3.2.13, we know that in order to have interesting topological cyclic homology (and Chern character), we should consider a smooth subalgebra $\mathcal{B} \subset C_r^*\Gamma$. Such an algebra exists and can be defined as follow:

DEFINITION 5.2.16 (§8.8, [**45**]). The *Connes-Moscovici algebra* is smooth and defined as the subalgebra:

$$\mathcal{B} := \{T \in C_r^*\Gamma \mid \forall k \in \mathbb{N},\ \delta^k(T) \in B(\ell_2(\gamma))\}$$

where $\delta(T) = [D, T]$, D being the unbounded operator on $\ell_2(\Gamma)$ defined for the standard orthonormal basis $(e_\gamma)_{\gamma\in\Gamma}$ of $\ell_2(\Gamma)$ as $De_\gamma = \|\gamma\| e_\gamma$. Here, $\|\cdot\|$ is a word metric on Γ.

REMARK 5.2.17 (§8.8, [**45**]). Then there exists an isomorphism identifying $K_*(\mathcal{B})$ with $K_*(C_r^*\Gamma)$ and the image of $\mathrm{ind}(\mathscr{D}_{(M,r)}) \in K_*(C_r^*\Gamma)$ in $K_*(\mathcal{B})$ is said to be the *smoothing of the index class*. In practice, we can achieve such smoothing directly, i.e. by replacing $C_r^*\Gamma$ by the smoothing subalgebra $\mathcal{B}$ in the construction above. Set $\mathcal{V}^\infty := \mathcal{B} \times_\Gamma \widetilde{M}$ and $\mathcal{E}^\infty := E \otimes \mathcal{V}^\infty$ for a Hermitian Clifford module $E \to M$. Then we analogously define $\mathcal{B}$-linear Dirac operator:

$$\mathscr{D}^\infty_{(M,r)} : C^\infty(M, E \otimes \mathcal{V}^\infty) \to C^\infty(M, F \otimes \mathcal{V}^\infty), \tag{5.2.2}$$

that we still denote $\mathscr{D}_{(M,r)}$. Then it is possible to define a pseudodifferential calculus as in §5.2, with $C_r^*\Gamma$ replaced by $\mathcal{B}$.

When restricting to a smooth subalgebra, some extra care has to be used for the spectral sections, since it is not at all obvious that a spectral section could be chosen in $\Psi^0_{\mathcal{B}}(\partial M, \mathcal{E}')$. However, this is possible for the following class of groups.

DEFINITION 5.2.18 (§8.11, [**45**]). Γ is called *virtually nilpotent* if it contains a nilpotent subgroup of finite index. Then Γ is of polynomial growth with respect to a (and thus any) word metric and the smooth subalgebra in this case corresponds to $\{f : \Gamma \to \mathbb{C} \mid \sup_{\gamma\in\Gamma} (1 + \|\gamma\|)^n |f(\gamma)|, \forall n \in \mathbb{N}\}$.

THEOREM 5.2.19 (Theorem 2.7, [**41**]). Let Γ be virtually nilpotent. Then a spectral section $\mathscr{P} \in \Psi^0_{C_r^*\Gamma}(\partial M, \mathcal{E}')$ can be chosen in $\Psi^0_{\mathcal{B}}(\partial M, \mathcal{E}')$.

In particular, the spectral section can be chosen to be symmetric if the assumption is satisfied. As a consequence, there is a well defined index class $\mathrm{ind}(\mathscr{D}_{(M,r)}, \mathscr{P}) \in K_0(\mathcal{B}) \cong K_0(C_r^*\Gamma)$ (Theorem 7.6, [**41**]).

5.3. Novikov's higher signatures as characters of a LogHQFT

DEFINITION 5.3.1 (§8.11, [**45**]). Let Γ be finitely generated. We say that Γ has *the extension property* if there exists a smooth subalgebra $\mathcal{B} \subset C_r^*\Gamma$ such that every $[c] \in H^*(B\Gamma, \mathbb{C})$ defines a cyclic cocycle $\varphi_c \in HC^*(\mathbb{C}\Gamma)$ which also extends to a continuous cyclic cocycle in $HC^*(\mathcal{B})$.

REMARK 5.3.2. Virtually nilpotent groups have the extension property. We have already seen that they are important for having well-defined spectral sections in $\Psi^0_{\mathcal{B}}$, hence we will consider Γ to be virtually nilpotent from now on.

Let Γ be a virtually nilpotent group and $\mathcal{B}$ the smooth subalgebra of $C_r^*\Gamma$. We consider the strict functor $F_\Gamma^{-\infty} : \mathbf{HCob}_n(B\Gamma)^* \to \mathbb{C}\text{-}\mathbf{Alg}$ defined as:

$$F_\Gamma^{-\infty}(M,r) := \Psi_{\mathcal{B}}^{-\infty}(M, \Lambda(M) \otimes \mathcal{B}) \qquad \forall (M,r) \in \mathrm{obj}(\mathbf{HCob}_n(B\Gamma)).$$

As in §4.4, we can define as insertion maps the algebra morphisms $\eta_{(N,s)}(M,r) : F_\Gamma^{-\infty}((M,r)) \hookrightarrow F_\Gamma^{-\infty}((M,r) \sqcup (N,s))$ as:

$$\eta_{(N,s)}(M,r)(\mathscr{A}) = j_{(N,s)}^* \circ \mathscr{A} \circ i_{(N,s)}^*, \qquad \mathscr{A} \in F_\Gamma^{-\infty}((M,r)),$$

where $j_{(N,s)} : (M,r) \sqcup (N,s) \to (M,r)$ and $i_{(N,s)} : (M,r) \hookrightarrow (M,r) \sqcup (N,s)$ are the projection and the inclusion, respectively. If $\widetilde{\eta}_{(N,s)} := K_0(\eta_{(N,s)})$, then by Lemma 3.2.7 we have once again that $F_\Gamma^{-\infty}$ is a a non-injective higher pretracial monoidal product representation and $(K_0(F_\Gamma^{-\infty}(\mathbf{HCob}_n(B)^*)), \widetilde{\eta}_{\sqcup \cdot}^k)$ is a presimplicial set. Clearly, $F_\Gamma^{-\infty}(M,r)$ is unoriented (as $F^{-\infty}(M)$ and $F_{\mathrm{vert}}^{-\infty}(\mathcal{M})$).

Lemma 5.3.3. $\Psi_{\mathcal{B}}^{-\infty}(M, E \otimes \mathcal{B})$ and $\mathcal{B}$ are Morita equivalent.

Proof. This is analogous to Example 3.2.16 and Lemma 4.4.1. In fact, by Schwarz Kernel Theorem, $\Psi_{\mathcal{B}}^{-\infty}(M, E \otimes \mathcal{B})$ is locally given by smooth functions in the matrix algebra $M_k(\mathcal{B})$.

□

Corollary 5.3.4. $K_0(F_\Gamma^{-\infty}((M,r)) \cong K_0(\mathcal{B}) \cong K_0(F_\Gamma^{-\infty}((M,r) \sqcup (N,s)))$. In particular, $\widetilde{\eta}_{\sqcup}^k$ are isomorphisms.

Moreover, a diffeomorphism $\phi : (M,r) \to (N,s)$ induces a canonical continuous isomorphism of algebras $\phi_\sharp : F_\Gamma^{-\infty}(M,r) \to F_\Gamma^{-\infty}(N,s)$ and pushes-down to a canonical linear isomorphism $\widetilde{\phi}_\sharp : K_0(F_\Gamma^{-\infty}(M,r)) \to K_0(F_\Gamma^{-\infty}(N,s))$, hence independent of the initial ϕ.

Proof. This follows because isomorphic algebras are in particular Morita equivalent.

□

Consider (M,r) of dimension $2m$ and the twisted signature operator $\mathscr{D}_{(M,r)}^{\mathrm{Sign}}$ associated to it. In order to obtain homotopy invariant index and Chern classes, we need symmetric spectral section for $\mathscr{D}_{(\partial M, r_{|\partial M})}^{\mathrm{Sign}}$, like in the family case. Their existance, in this case, requires the following assumption (Assumption (H2), [**42**]):

Definition 5.3.5 (*Middle-degree assumption*). Let $\dim M = 2m$. If d is the de Rham differential on $\widetilde{\partial M}$, endowed with a Γ-invariant metric, then we assume that $dd^* + d^*d$ acting on $L^2(\widetilde{\partial M}, \Lambda^m(\widetilde{\partial M}))/\ker(d)$ has a gap at zero, i.e.

$$\mathrm{spec}(dd^* + d^*d) \cap (-\delta, \delta) = \{0\}.$$

REMARK 5.3.6. The middle degree assumption is analogous to the condition that $\ker(\Delta^{\mathrm{sign}}_{m,Y_b})$ has constant dimension with respect to $b \in B$ (Proposition 4.4.5).

PROPOSITION 5.3.7 (Proposition 4.4, **[42]**). Let us assume Definition 5.3.5. Then there exist symmetric spectral sections, such that for any two such sections $\mathscr{P}, \mathscr{Q} \in \Psi^0_{C^*_r\Gamma}(\partial M, \mathcal{E}')$,

$$[\mathscr{P} - \mathscr{Q}] = 0 \quad \text{in} \quad K_0(C^*_r\Gamma) \otimes \mathbb{C}.$$

Let $\mathscr{P}$ be a symmetric spectral section and define the following universal LogTQFT:

$$u\text{-}\log^{\mathrm{Sign}} : \mathcal{N}\mathbf{HCob}_n(B\Gamma) \to K_0(F^{-\infty}_\Gamma(\mathbf{HCob}_n(B\Gamma)^*)) \otimes \mathbb{C}$$

by setting as a map on 1-simplices:

$$u\text{-}\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)} : \mathrm{mor}((M_0,r_0),(M_1,r_1)) \to K_0(F^{-\infty}_\Gamma((M_0,r_0)\sqcup(M_1,r_1))) \otimes \mathbb{C}$$

$$u\text{-}\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)}(W,F) := \widetilde{\phi}_{\sharp,(M_0,r_0)\sqcup(M_1,r_1)}\left(\mathrm{ind}(\mathscr{D}_{(W,F)},\mathscr{P})\right), \tag{5.3.1}$$

with $\widetilde{\phi}_{\sharp,(M_0,r_0)\sqcup(M_1,r_1)} : K_0(F^{-\infty}_\Gamma(\partial\mathcal{W})) \otimes \mathbb{C} \to K_0(F^{-\infty}_\Gamma((M_0,r_0)\sqcup(M_1,r_1))) \otimes \mathbb{C}$ the canonical isomorphism.

THEOREM 5.3.8. With respect to gluing, we have :

$$\begin{aligned}\widetilde{\eta}_{(M_1,r_1)} u\text{-}\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_2,r_2)}(W,F) = \\ = \widetilde{\eta}_{(M_2,r_2)} u\text{-}\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)}(W_1,F_1) + \widetilde{\eta}_{(M_0,r_0)} u\text{-}\log^{\mathrm{Sign}}_{(M_1,r_1)\sqcup(M_2,r_2)}(W_2,F_2)\end{aligned}$$

in $K_0(F^{-\infty}_\Gamma((M_0,r_0)\sqcup(M_1,r_1)\sqcup(M_2,r_2)))\otimes\mathbb{C}$. Therefore (5.3.1) defines a universal LogTQFT.

PROOF. The $\widetilde{\eta}_{(M_i,r_i)}$ are isomorphisms into

$$K_0(F^{-\infty}_\Gamma((M_0,r_0)\sqcup(M_1,r_1)\sqcup(M_2,r_2))) \otimes \mathbb{C} \cong K_0(\mathcal{B}) \otimes \mathbb{C},$$

where Corollary 5.2.14 holds.

$\square$

COROLLARY 5.3.9. By composition with $\mathrm{ch}_* : K_0(\mathcal{B}) \to HC_{2*}(\mathcal{B})$ we obtain the following LogHQFT:

$$\log^{\mathrm{Sign}} : \mathcal{N}\mathbf{HCob}_n(B\Gamma) \to HC_{2*}(F_\Gamma^{-\infty}(\mathbf{HCob}_n(B\Gamma)^*))$$

$$\log^{\mathrm{Sign}}_{(M_0,r_0)\sqcup(M_1,r_1)}(W,F) := \mathrm{ch}_*\left(\mathrm{ind}(\mathscr{D}_{(W,F)},\mathscr{P})\right).$$

REMARK 5.3.10. $\mathrm{ch}_*(\mathrm{ind}(\mathscr{D}_{(M,r)},\mathscr{P}))$ actually belongs to $\widehat{H}_*(\mathcal{B})$ (Theorem 6.3, [**42**]), the *noncommutative topological de Rham homology* of $\mathcal{B}$. However, this is contained in the cyclic homology of $\mathcal{B}$. We refer to the paragraphs §8.4 – §8.7 of [**45**] for the definition of noncommutative topological de Rham homology and its relationship with cyclic homology.

For $[c] \in H^*(\Gamma,\mathbb{C})$, let $\varphi_c \in HC^*(\mathcal{B})$ be its associated cyclic cocycle. From §3.1 we know that a higher trace $\tau^c : HC_*(\mathcal{B}) \to \mathbb{C}$ can be defined by Kronecker pairing with φ_c.

PROPOSITION 5.3.11. The right-hand side of (5.3.1) depends only on the oriented homotopy class (W,F). Also, it has log-character:

$$\tau^c_{(M_0\sqcup M_1,s_1\sqcup s_2)}\left(\log^{\mathrm{Sign}}_{(M_0\sqcup M_1,s_1\sqcup s_2)}(W,r)\right) = \mathrm{Sign}(W,r;c),$$

where $\mathrm{Sign}(W,r;c)$ is a Novikov's higher signature. Log-additivity clearly yields additivity of Novikov's higher signatures.

PROOF. By definition of higher trace:

$$\tau^c_{(M_0\sqcup M_1,s_1\sqcup s_2)}\left(\log^{\mathrm{Sign}}_{(M_0\sqcup M_1,s_1\sqcup s_2)}(W,r)\right) = \langle \mathrm{ch}(\mathrm{ind}(\mathscr{D}^{\mathrm{Sign}},\mathscr{P})),\varphi_c\rangle =: \mathrm{Sign}(W,r;c), \tag{5.3.2}$$

which is the definition of the Novikov signature associated to c.

The first statement is a consequence of the fact that Novikov signatures are homotopy invariants when Γ is nilpotent and the middle-degree assumption (Definition 5.3.5) holds.

□

Indeed, (5.3.2) is the definition of higher signatures for a manifold with boundary. The closed case is very similar:

DEFINITION 5.3.12 (§5.2, [**45**]). The *Novikov's higher signature of* $(M,r) \in \mathrm{obj}(\mathbf{HCob}_n(B\Gamma))$ associated to $[c] \in H^*(B\Gamma,\mathbb{R})$ is:

$$\mathrm{sign}(M,r;[c]) := \int_M [L(M)] \wedge r^*[c] = \langle L(M) \cup r^*[c],[M]\rangle.$$

REMARK 5.3.13. If $\dim M = 4k$ and $[c] = 1$ we obtain

$$\operatorname{sign}(M, r; 1) := \langle L(M), [M] \rangle := \int_M L(M) = \sigma(M) \in \mathbb{Z}, \tag{5.3.3}$$

i.e. the topological signature of M.

Thus, if $c = 1$, then $\tau^1_{(M_0 \sqcup M_1, s_1 \sqcup s_2)} \left(\log^{\mathrm{Sign}}_{(M_0 \sqcup M_1, s_1 \sqcup s_2)}(\overline{W}, r) \right) = \langle \mathrm{ch}(\mathrm{ind}(\mathscr{D}^{\mathrm{Sign}}, \mathscr{P})), \varphi_1 \rangle =: \mathrm{Sign}(W, r; 1) = \sigma(W)$.

Part 3

Logarithms and torsion invariants

CHAPTER 6

Torsion invariants

In this chapter we will study an exotic torsion invariant of manifolds, which we defined via the residue trace. It is similar in nature to the analytic torsion and as such is a generalized log-determinant that can be represented in the functorial framework of LogTQFTs.

We start with a survey of Reidemeister and analytic torsion, since their construction will highlight the steps that led us to define exotic torsions. We will represent the analytic torsion as a trace-character of a *torsion logarithm* and will be able to define a *residue* torsion by composition with the residue trace. We will generalize our results to fibre bundles (with closed fibre), and manifolds with boundary and relative/absolute boundary conditions.

Along the way, we will study a topological invariant called *secondary* Euler characteristic, that arises from the definition of residue torsion.

6.1. Reidemeister Torsion

6.1.1. The Torsion of a Matrix. The definitions and results in this section are taken from [**15**], unless otherwise stated.

Let $GL_n(R)$ be the n^{th} general linear group with coefficients in a ring R with unit 1_R and, for $i \neq j$, let $E^n_{i,j}$ be the $n \times n$ matrix with coefficient $e_{ij} = 1_R$ and 0_R elsewhere.

Definition 6.1.1. Let I^n be the $n \times n$ identity matrix and $c \in R$. Then matrices of the form $I^n + cE^n_{i,j}$, for some $n \in \mathbb{N}$, are called *elementary*.

Remark 6.1.2. Let $E(R)$ be the subgroup of $GL(R) = \varinjlim GL_n(R)$ generated by the elementary matrices. Then $E(R) \trianglelefteq GL(R)$ (i.e. is normal) and it coincides with the commutator subgroup $GL(R)' = [GL(R), GL(R)]$ (defined in Lemma 1.2.5).

Remark 6.1.3. Let us consider the quotient group $GL(R)/E(R)$, which is defined by similarity, i.e.

'$A \sim B$ if and only if there exist $E_1, E_2 \in E(R)$ such that $A = E_1 B E_2$'.

Then, for every normal subgroup $E(R) \trianglelefteq H \trianglelefteq GL(R)$, the quotient $GL(R)/H$ is abelian.

For R^* the subring of units, let $G \leq R^*$, i.e. a subgroup, and $c \in G$. We consider the diagonal matrices of the form $I^n + (c - 1_R)E^n_{i,i}$ with $1 \leq i \leq n$, i.e. diagonal matrices with 1_R entries everywhere but in the (i,i)-position, which is c. Let $E_G(R) \leq GL(R)$ be the subgroup generated by such matrices and $E(R)$.

Definition 6.1.4. If $\tau_G : GL(R) \to K_G(R)$ denotes the canonical projection onto the quotient $K_G(R) := GL(R)/E_G(R)$, then the *torsion of the matrix A* is the class $\tau_G(A)$.

Example 6.1.5 (§2, [**53**]). If $G = \{1\}$, $K_1(R) := GL(R)/E(R)$ is called *Whitehead group of R*, while $\overline{K}_1(R) := K_G(R)$ for $G = \{-1, 1\}$ is called *reduced Whitehead group of R.*

If we assume R commutative, then we can represent the torsion in terms of the determinant $\det : GL(R) \to R^*$ as follows. Recall that the determinant is a surjective homomorphism with kernel $SL(R) := \{A \in GL(R) |\ \det(A) = 1\}$.

Proposition 6.1.6. Let R be commutative, $G \leq R^*$, and $SK_G(R) := \tau_G(SL(R))$. Then there is a short exact sequence:

$$0 \longrightarrow SK_G(R) \longrightarrow K_G(R) \xrightarrow{\widetilde{\det}} R^*/G \longrightarrow 0$$

which is split $s : R^*/G \to K_G(R)$, where $s(rG) = \tau_G(r)$.

Corollary 6.1.7. If R is a field, then $\widetilde{\det} : K_G(R) \to R^*/G$ is an isomorphism and the torsion $\tau_G : GL(R) \to K_G(R)$ can be identified with the matrix determinant modulo G.

Example 6.1.8. Let $R = \mathbb{R}$ and $G = \{-1, 1\}$. Then $\tau_G(A) \in \overline{K}_1(\mathbb{R})$ can be identified with $|\det(A)|$ and $\overline{K}_1(\mathbb{R}) \cong \mathbb{R}^+$.

Remark 6.1.9. Unlike in [**15**] and [**53**], which use an *additive* notation, we will keep the *multiplicative* one[1], as in [**59**], since we will define Reidemeister torsion in terms of the determinant. An additive formalism will naturally arise though composition with the real logarithm. Therefore, we will write $\tau_G(AB) = \tau_G(A)\tau_G(B)$.

[1] Usually, the notation is additive when working with abelian groups.

6.1.2. The Reidemeister torsion of a chain complex.

REMARK 6.1.10. From now on, we will assume that $-1_R \in G \leq R^*$.

DEFINITION 6.1.11 (§12, [**15**]). Let $f : V \to W$ be an isomorphism of finitely generated free R-modules, for R a commutative ring. Let v and w be bases for V and W, respectively, and let A_f denote the (invertible) matrix associated to f. Then the torsion of $f : V \to W$ is $\tau_G(f) := \tau_G(A_f)$.

REMARK 6.1.12 (§2, [**53**]). $\tau_G(f)$ of Definition 6.1.4 *does* depend on the chosen bases.

Since $f : V \to W$ is a short exact sequence, one can generalize the previous definition to chain complexes. Thus, let us consider a (finite) chain complex of based finitely generated free R-modules:

$$C : 0 \longrightarrow C_N \xrightarrow{d} \cdots \xrightarrow{d} C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} \ldots C_1 \xrightarrow{d} C_0 \longrightarrow 0.$$

Set $Z_r := \ker(d : C_r \to C_{r-1})$ and $B_r := \operatorname{ran}(d : C_{r+1} \to C_r)$, so that $H_r(C) = Z_r/B_r$ will denote the homology R-modules of (C, d).

DEFINITION 6.1.13 ([**64**]). A chain complex (C, d) is called *acyclic* if $\forall r \geq 0$ $H_r(C) = 0$, i.e. the sequence is exact.

PROPOSITION 6.1.14 ((13.1), [**15**]). If (C, d) is acyclic, then there exists a degree-one module homomorphism $\delta : C \to C$, i.e. a collection of homomorphisms $\delta : C_r \to C_{r+1}$, such that $\delta d + d\delta = 1_C$, the identity chain map. For any such δ, we have $d\delta_{|B_{r-1}} = \mathrm{id}$ and thus $C_r = B_r \oplus \delta B_{r-1}$, $\forall r \geq 0$.

REMARK 6.1.15 ([**64**]). The chain map $\delta : C \to C$ of Proposition 6.1.14 is called *chain contraction* and is a chain homotopy between 1_C and the zero chain map $0_C : C \to C$. Moreover, the previous Proposition yields that $d_{|\delta B_{r-1}} : \delta B_{r-1} \to B_{r-1}$ is an isomorphism.

LEMMA 6.1.16 (Lemma 3, [**64**]). Let (C, d) be an acyclic R-module chain complex and $\delta : C \to C$ a chain contraction. Then the R-module morphism:

$$(d + \delta)_{|C_{\mathrm{odd}}} : C_{\mathrm{odd}} \to C_{\mathrm{even}}$$

is an isomorphism, where $C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus \ldots$ and $C_{\mathrm{even}} = C_0 \oplus C_2 \oplus \ldots$.

PROPOSITION 6.1.17 ((14,2), [**15**]). (C, d) and the 'wrapped up' exact complex:

$$C' : 0 \longrightarrow C_{\text{odd}} \xrightarrow{d+\delta} C_{\text{even}} \longrightarrow 0$$

are *stably equivalent*, i.e. C and C' are isomorphic modulo trivial complexes[2].

This latter result motivates the following:

DEFINITION 6.1.18 (§15, [**15**]). Let (C, d) be an acyclic R-module chain complex. Then the *torsion of* C is defined as $\tau_G(C) := \tau_G((d+\delta)_{|C_{\text{odd}}}) \in K_G(R)$ and is independent of the chain contraction δ ((15.3), [**15**]).

REMARK 6.1.19. As anticipated in Remark 6.1.21, $\tau_G(C)$ depends on a choice of basis for C. Let c_r be a basis for C_r; then $c_{\text{odd}} = \bigoplus_{j\geq 0} c_{2j+1}$, $c_{\text{even}} = \bigoplus_{j\geq 0} c_{2j}$, and $c = \bigoplus_{j\geq 0} c_j$ are bases for C_{odd}, C_{even}, and C, respectively, with respect to which the isomorphism $(d+\delta)_{|C_{\text{odd}}} : C_{\text{odd}} \to C_{\text{even}}$ can be represented by the non-singular square matrix (denoted with the same symbol):

$$(d+\delta)_{|C_{\text{odd}},c} = \begin{pmatrix} d & 0 & 0 & \dots \\ \delta & d & 0 & \dots \\ 0 & \delta & d & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

PROPOSITION 6.1.20 ((15.1), [**15**]). With respect to a basis c for C, we have $\tau_G((d+\delta)_{|C_{\text{odd}},c}) = \tau_G((d+\delta)_{|C_{\text{even}},c})^{-1}$.

If c'_r be another basis for C_r, let (c'_r/c_r) represent the matrix of the change of basis $c_r \mapsto c'_r$. Thence,

$$(c'_{\text{odd}}/c_{\text{odd}}) = \bigoplus_{j\geq 0}(c'_{2j+1}/c_{2j+1}) \qquad \text{and} \qquad (c'_{\text{even}}/c_{\text{even}}) = \bigoplus_{j\geq 0}(c'_{2j}/c_{2j}).$$

PROPOSITION 6.1.21. Let c, c' be two arbitrary bases for the acyclic R-module chain complex (C, d). Then, for $\tau_G(C, c) := \tau_G((d+\delta)_{|C_{\text{odd}},c})$:

$$\tau_G(C, c') = \tau_G(C, c) \cdot \prod_{r\geq 0} \tau_G(c_r/c'_r)^{(-1)^r}.$$

In general, for a short exact sequence of chain complexes, the torsion is multiplicative:

[2]These are complexes whose boundary map can be represented by the identity matrix for a particular choice of basis. See §14, [**15**], for a detailed presentation.

THEOREM 6.1.22 (Theorem 3.1, [**53**]). Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of (finite) chain complexes of finitely generated free R-modules. Let c, c', and c'' be bases for C, C', and C'', respectively, such that the matrix of change of basis $(c'c''/c)$ belongs to $E_G(R)$, and let $\mathcal{H}$ denote the exact homology sequence of the homology groups of C', C, and C''. Then:

$$\tau_G(C) = \tau_G(C')\tau_G(C'')\tau_G(\mathcal{H}).$$

In particular, if they are all acyclic[3], $\tau_G(C) = \tau_G(C')\tau_G(C'')$.

DEFINITION 6.1.23. Let $\mathbb{F}$ be a field and $G = \{-1, 1\}$. Then the torsion $\tau(C, c) := \tau_G(C, c) \in \overline{K}_1(\mathbb{F})$ of an acyclic $\mathbb{F}$-module chain complex C is called *Reidemeister-Franz torsion* (or *R-torsion*) of C.

REMARK 6.1.24 (§18, [**15**]). In general, the R-torsion arises after a suitable change of rings. In fact, for $\rho : R \to S$ a change of rings such that $\rho(G) \leq G'$, for $-1 \in G' \leq S^*$, one obtains a new complex C_ρ out of C, which can be acyclic even if C is not, and a new algebraic invariant: the torsion $\tau_{G'}(C_\rho) \in K_{G'}(S)$.

REMARK 6.1.25. By Corollary 6.1.7, R-torsion can equivalently be defined as the determinant:

$$\tau(C, c) := \det(d + \delta : C_{\text{odd}} \to C_{\text{even}}) \in \mathbb{F}, \qquad \text{for } c \text{ a basis of } C.$$

DEFINITION 6.1.26 (Definition 14, [**64**]). Let B_r be free for each $r \geq 0$, and b_r a basis. An *internal basis* of C is a basis obtained extending b_r to the whole C_r via the isomorphism $d_{|\delta B_{r-1}} : \delta B_{r-1} \to B_{r-1}$ of Remark 6.1.15.

PROPOSITION 6.1.27 (§3, [**53**]). For $b = \bigoplus_{r \geq 0}(b_r, \delta b_{r-1})$ an internal basis, $\tau(C, b) = 1$ and:

$$\tau_G(C, c) = \prod_{r \geq 0} \tau_G(b_r, \delta b_{r-1}/c_r)^{(-1)^r}. \tag{6.1.1}$$

In particular, $\tau_G(C, c)$ does not depend on the particular internal basis b chosen.

REMARK 6.1.28 (§, [**53**]). Milnor defines torsion exactly as (6.1.1) and in this way, he can define torsion for stably free modules B_r and for a non-acyclic chain complex (C, d). In fact, if h_r is a basis for $H_r(C)$, then b_r, h_r and b_{r-1} form a

[3]It actually suffices that C and one between C' and C'' are acyclic.

basis for C_r. If we denote the matrix of change of basis with $(b_r, h_r, \delta b_{r-1}/c_r)$, then Milnor's definition of torsion for non-acyclic complexes is:

$$\tau_G(C,c,h) = \prod_{r\geq 0} \tau_G(b_r, h_r, \delta b_{r-1}/c_r)^{(-1)^r},$$

which depends on the bases c and $h = \bigoplus_{r\geq 0} h_r$.

REMARK 6.1.29. If we combine Corollary 6.1.7 and Example 6.1.8, we obtain Ray and Singer's definition of R-torsion of an acyclic complex in [**65**]:

$$\tau(C,c) = \prod_{r\geq 0} |\det(b_r, \delta b_{r-1}/c_r)|^{(-1)^r} \in \mathbb{R}^+,$$

and therefore, $\log \tau(C,c) = \sum_{r\geq 0} (-1)^r \log|\det(b_r, \delta b_{r-1}/c_r)|$.

6.1.3. Reidemeister torsion of manifolds. Given a CW-complex, one can associate to it an acyclic chain complex and hence an R-torsion, which represents a secondary topological invariant of the CW-complex, i.e. a topological invariant defined at the level of the chain complex, which can therefore distinguish between spaces with same homology and fundamental groups (such as Lens spaces, [**67**]).

First, let us see how to define the R-torsion of a finite and connected CW-complex $X = \bigcup_{r=0}^n \bigcup e^r$, with $e^r \subset X$ an r-cell, as shown in [**64**] and [**65**]. Let $\widetilde{X} = \bigcup_{g\in\pi_1(X)} \bigcup_{r=0}^n \bigcup g\tilde{e}^r$ be the universal cover of X, where $\tilde{e}^r$ is a lift of the cell e^r and $\pi_1(X)$ is acting on $\widetilde{X}$ as deck transformation group, i.e.

$$\pi_1(X) \times \widetilde{X} \to \widetilde{X};\ (g,x) \mapsto gx.$$

Let $X^{(r)} = \bigcup_{j\leq r} \bigcup e^r$ be the r-skeleton of X, with preferred basis given by the cells of $X^{(r)}$ and induced cover $\widetilde{X}^{(r)}$, and consider the relative homology modules $C_r(\widetilde{X}) := H_r(\widetilde{X}^{(r)}, \widetilde{X}^{(r-1)})$ and the group ring $\mathbb{R}[\pi_1(X)]$ of finite formal sums $\sum_k \alpha_k g_k$, for $\alpha_k \in \mathbb{R}$ and $g_k \in \pi_1(X)$. Then $C_r(\widetilde{X})$ is a based finitely generated free $\mathbb{R}[\pi_1(X)]$-module generated by the e^r cells, and fits into the cellular chain complex

$$C(\widetilde{X}) : C_n(\widetilde{X}) \xrightarrow{d} C_{n-1}(\widetilde{X}) \xrightarrow{d} \cdots \xrightarrow{d} C_1(\widetilde{X}) \xrightarrow{d} C_0(\widetilde{X}),$$

where d is the boundary operator induced by the natural boundary operator of the CW-complex. With respect to a preferred basis, d is represented by a matrix with $\mathbb{R}[\pi_1(X)]$ entries.

However, this construction does not provide an acyclic complex, as $H_0(C(\widetilde{X})) = H_0(\widetilde{X}) = \mathbb{R}$. Therefore, as suggested in Remark 6.1.24, we can consider a *representation* of $\pi_1(X)$, i.e. a group homomorphism $\rho : \pi_1(X) \to O(N)$, such that we can construct a new complex which is also acyclic. We remark here that, since ρ takes values in $O(N)$, this will remove every possible ambiguity in the definition of the R-torsion (see **[59]**, for instance, for a general treatment of ambiguities of the definition of R-torsion). A representation ρ extends to a ring homomorphism for $\mathbb{R}[\pi_1(X)]$, thus making $\mathbb{R}^n$ into a right $\mathbb{R}[\pi_1(X)]$-module. The associated new complex, denoted $(C(X,\rho),d)$, is defined as the chain complex of finitely generated free modules

$$C_r(X,\rho) := \mathbb{R}^N \otimes_{\mathbb{R}[\pi_1(X)]} C_r(\widetilde{X}). \tag{6.1.2}$$

Moreover, a preferred basis of $C_r(X,\rho)$ is realised by the equivalence class of $(\tilde{e}^r, v)$ modulo the relation $(\tilde{e}^r, v) \sim (g \cdot \tilde{e}^r, \rho(g^{-1})v)$, with $v \in \mathbb{R}^N$ and $g \in \pi_1(X)$. The boundary operator d is the one induced on the equivalence classes by the one on $C(\widetilde{X})$, i.e. $d[\tilde{e}^r, v] = [de^r, v]$ (see §5.3.1, **[67]**).

Definition 6.1.30 (Definition 1.3, **[65]**). Let $\rho : \pi_1(X) \to O(N)$ be a ring homomorphism such that $C(X,\rho)$ is acyclic. Then the Reidemeister torsion of X is defined as $\tau_X(\rho) := \tau(C(X,\rho))$. Here, the dependence on a preferred basis is omitted from the notation.

With respect to a basis for each $C_r(X,\rho)$, $d_r : C_r(X,\rho) \to C_{r-1}(X,\rho)$ is represented by a real matrix. Let $d_r^* : C_{r-1}(X,\rho) \to C_r(X,\rho)$ be its transpose.

Definition 6.1.31 (§1, **[65]**). The matrix $\Delta_r^c := d_{r+1}d_{r+1}^* + d_r^* d_r$, which acts on $C_r(X,\rho)$, is called *the combinatorial Laplacian*.

Proposition 6.1.32 (Proposition 1.7, **[65]**). Let $\tau_X(\rho)$ be the R-torsion of a finite CW-complex $X = \bigcup_{r=0}^n \bigcup e^r$, with $\rho : \pi_1(X) \to O(N)$ an acyclic representation. Then:

$$\log \tau_X(\rho) = \frac{1}{2}\sum_{r=0}^{n} (-1)^{r+1} r \log\det \Delta_r^c. \tag{6.1.3}$$

Remark 6.1.33. Let λ_i be an eigenvalue of Δ_r^c, which is positive since Δ_r^c is positive definite (see the proof of Proposition 1.7, **[65]**). Then the sum $\sum_{\lambda_i>0} \lambda_i^{-s}$, $s \in \mathbb{C}$, is holomorphic for $\Re(s)$ large enough and defines a *spectral zeta function* for Δ_r^c as the meromorphic extension

$$\zeta_r^c(s) := \zeta(\Delta_r^c, s) = \sum_{\lambda_i>0} \lambda_i^{-s}|^{\mathrm{mer}}.$$

In particular, $\zeta_r^c(s)$ is holomorphic at $s = 0$ and:

$$\frac{d}{ds}\zeta_r^c(s)_{|s=0} = -\sum_{\lambda_i} \log \lambda_i = -\log \prod_{\lambda_i} \lambda_i = -\log \det \Delta_r^c. \tag{6.1.4}$$

Hence, (6.1.3) can be equivalently written:

$$\log \tau_X(\rho) = \frac{1}{2}\sum_{r=0}^{n}(-1)^r r \; \frac{d}{ds}\zeta_r^c(0). \tag{6.1.5}$$

LEMMA 6.1.34 (Combinatorial invariance of R-torsion; Lemma 7.1, [**53**]). $\tau_X(\rho)$ is invariant under subdivision of X. Hence, it is a combinatorial invariant of X.

THEOREM 6.1.35 (Topological invariance of R-torsion; [**14**]). Let $f : X_1 \to X_2$ be a homeomorphism of CW-complexes. Then $\tau_{X_1}(\rho f_*) = \tau_{X_2}(\rho)$.

Finally, let M be an n-dimensional manifold with possibly $\partial M \neq \emptyset$. Then M admits a C^1-triangulation X and thus:

DEFINITION 6.1.36 (§9, [**53**]). Let M be a manifold with C^1-triangulation X. Then the R-torsion of M is the scalar $\tau_M(\rho) := \tau_X(\rho)$.

REMARK 6.1.37. $\tau_M(\rho)$ does not depend on the C^1-triangulation of M, but only on the manifold M and the representation ρ (Lemma 9.1, [**53**]). In particular, from Theorem 6.1.35 we have that the R-torsion is a topological invariant of a manifold.

THEOREM 6.1.38 (§6, [**55**]). Let M be a closed oriented manifold, $\dim M = n$ even. Then $\log \tau_M(\rho) = 0$.

Let now $Y \subseteq X$ be a subcomplex. The construction for X applies now also to the pair (X, Y) (see §8 in [**53**]), thus there exists a chain complex of finitely generated free $\mathbb{R}[\pi_1(X)]$-modules $C(\widetilde{X}, \widetilde{Y})$, with $\widetilde{X} \overset{p}{\to} X$ the universal cover and $\widetilde{Y} := p^{-1}(Y)$. We observe that the inclusion $\iota : Y \hookrightarrow X$ defines an homomorphism $\iota_* : \pi_1(Y) \to \pi_1(X)$, which yields a representation for $\pi_1(Y)$ once it is composed with $\rho : \pi_1(X) \to O(N)$. Thence it is possible to define a relative chain complex $(C(X, Y, \rho), d)$, where $C_r(X, Y, \rho)$ is as in (6.1.2). See [**82**] for a detailed construction.

DEFINITION 6.1.39. Let $\rho : \pi_1(X) \to O(N)$ be a ring homomorphism such that $C(X, Y, \rho)$ is acyclic. Then the R-torsion of the CW-pair (X, Y) is defined as $\tau_{X,Y}(\rho) := \tau(C(X, Y, \rho))$.

As a consequence of Theorem 6.1.22, the three R-torsions $\tau_X(\rho)$, $\tau_{X,Y}(\rho)$ and $\tau_Y(\rho \circ \iota_*)$ relate through the following result:

THEOREM 6.1.40 (0.2.2, [**82**]). $\tau_X(\rho) = \tau_{X,Y}(\rho) \cdot \tau_Y(\rho \circ \iota_*)$.

It is therefore natural to extend the definition of relative R-torsion to manifolds with boundary:

DEFINITION 6.1.41 (§9, [**53**]). Let M be a manifold with non-empty boundary ∂M, and (X, Y) a CW-triangulation such that Y is a triangulation of ∂M. Then the *relative R-torsion* of M is $\tau_{M,\partial M}(\rho) := \tau_{X,Y}(\rho)$.

REMARK 6.1.42 (Remarks 2.12 & 2.62, [**59**]). If $\partial M \neq \emptyset$, then $\tau_M(\rho)$ is called *absolute R-torsion* of M. Moreover, Lemma 6.1.34 holds generally for CW-pairs (X, Y), [**53**]. Thus $\tau_{X,Y}(\rho)$ is invariant under subdivision and $\tau_{M,\partial M}(\rho)$ is independent of the triangulation (X, Y). In fact, $\tau_{M,\partial M}(\rho)$ is a smooth invariant, but not a topological invariant, in general.

We conclude with a gluing formulas for the R-torsion of CW-pairs, which is a direct consequence of Theorem 6.1.22:

THEOREM 6.1.43 (Gluing of relative R-torsion; Proposition 1.5, [**83**]). Let (X_i, Y_i), $i = 1, 2$, be two CW-pairs such that $X := X_1 \cup X_2$ and $N := X_1 \cap X_2$, with $N \cap Y_i = \emptyset\ \forall i = 1, 2$. Then, for $\iota : N \to X$, $\iota_i : X_i \to X$ the natural inclusions:

$$\tau_{X,Y_1 \sqcup Y_2}(\rho) = \tau_{X_1,Y_1 \sqcup N}(\rho\iota_{1*}) \cdot \tau_{X_2,Y_2 \sqcup N}(\rho\iota_{2*}) \cdot \tau_N(\rho\iota_*).$$

Combined with Theorem 6.1.40, we obtain:

COROLLARY 6.1.44 (Gluing of absolute R-torsion; Proposition 0.2.3, [**82**]). Let X_i, $i = 1, 2$, be two CW-complexes such that $X := X_1 \cup X_2$ and $N := X_1 \cap X_2$. Then, for $\iota : N \to X$, $\iota_i : X_i \to X$ the natural inclusions, we have:

$$\tau_X(\rho) = \tau_{X_1}(\rho\iota_{1*}) \cdot \tau_{X_2}(\rho\iota_{2*}) \cdot \tau_N(\rho\iota_*)^{-1}.$$

6.2. Analytic and Residue Torsion of a closed manifold

6.2.1. Analytic Torsion. The spectral zeta function definition of the R-torsion (6.1.5) motivated Ray and Singer to define in [**65**] an analytic counterpart as follows.

Let X be a closed oriented manifold and $\rho : \pi_1(X) \to O(N)$ an orthogonal representation. For $E_\rho := \widetilde{X} \times_\rho \mathbb{C}^N$ the principal (flat) bundle associated to ρ, let $\Omega(X, E_\rho)$ be the twisted de Rham bundle with coefficients in E_ρ and $\Delta_k : \Omega^k(X, E_\rho) \to \Omega^k(X, E_\rho)$ the Laplacian on twisted k-forms, as usual.

REMARK 6.2.1 ([**55**]). By Hodge Theorem, Δ_k has real non-negative eigenvalues with finite multiplicity, which accumulate at infinity, and $\ker(\Delta_k) \cong H^k(X)$, thus Δ_k is strictly positive if and only if $\Omega(X, E_\rho)$ is acyclic. Let λ_i be an eigenvalue of Δ_k and consider the sum $\sum_{\lambda_i>0} \lambda_i^{-s}$, $s \in \mathbb{C}$ as in Remark 6.1.33. Such sum is holomorphic for $\Re(s) > \frac{1}{2}\dim X$ and defines a spectral zeta function for Δ_k as the meromorphic extension $\zeta_{k,\rho}(s) := \zeta(\Delta_k, s) = \sum_{\lambda_i>0} \lambda_i^{-s}|^{\mathrm{mer}}$, which is holomorphic at $s = 0$.

DEFINITION 6.2.2 (Definition 1.6, [**65**]). Let X be a closed manifold and let $\rho : \pi_1(X) \to O(N)$ be an orthogonal representation. Then the *analytic torsion of* X is the scalar:

$$T_X(\rho) = \prod_{k=0}^{n} \exp\left((-1)^k \frac{k}{2} \frac{d}{ds} \zeta_{k,\rho}(0)\right),$$

i.e. $\log T_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k k \frac{d}{ds}\zeta_{k,\rho}(0)$.

REMARK 6.2.3 (§1, [**65**]). Let $\det_\zeta\ \Delta_k := e^{-\zeta'_{k,\rho}(0)}$ be the *zeta determinant* of Δ_k, a regularized extension of the determinant of a matrix. Then the analytic torsion can be equivalently written as:

$$\log T_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} k \log \det{}_\zeta \Delta_k. \tag{6.2.1}$$

We remark the similarity with Proposition 6.1.32.

In some cases, the analytic torsion is a smooth invariant:

THEOREM 6.2.4 (Theorem 2.1, [**65**]). Let X be a closed oriented Riemannian manifold and $\rho : \pi_1(X) \to O(N)$ an orthogonal representation. If $\Omega(X, E_\rho)$ is acyclic, then $T_X(\rho)$ is independent of the choice of Riemannian metric on X.

One may wonder why Definition 6.2.2 involves the weight k, or similarly, why we do not consider a torsion of the form $\log \widetilde{T}_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\log\det_\zeta \Delta_k$, in analogy with the Euler characteristic. The reason is that such *unweighed* version is trivial, as the following results show.

REMARK 6.2.5 (§5.3.2, [**67**]). For each $k = 0, \ldots, n$, Δ^k and Δ^{n-k} are *isospectral*, i.e. they have the same eigenvalues, since $*^k\Delta^k = \Delta^{n-k}*^k$. Therefore, $\forall s \in \mathbb{C}$, we obtain Poincaré Duality, i.e.

$$\zeta_{k,\rho}(s) = \zeta_{n-k,\rho}(s), \tag{6.2.2}$$

which yields:

PROPOSITION 6.2.6.

i) n even: $\frac{n}{2}\sum_{k=0}^{n}(-1)^k\ \zeta_{k,\rho}(s) = \sum_{k=0}^{n}(-1)^k k\ \zeta_{k,\rho}(s)$,

ii) n odd: $\sum_{k=0}^{n}(-1)^k\ \zeta_{k,\rho}(s) = 0$.

PROOF. *i)* If n is even, then $(-1)^{n-k} = (-1)^k$ and:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k\zeta_{k,\rho}(s) &\overset{(6.2.2)}{=} \sum_{k=0}^{n}(-1)^k k\zeta_{n-k,\rho}(s) = \sum_{k=0}^{n}(-1)^{n-k}(n-k)\zeta_{k,\rho}(s)\\ &= \sum_{k=0}^{n}(-1)^k(n-k)\zeta_{k,\rho}(s) = n\sum_{k=0}^{n}(-1)^k\zeta_{k,\rho}(s) - \sum_{k=0}^{n}(-1)^k k\zeta_{k,\rho}(s).\end{aligned}$$

ii) If n is odd, then $(-1)^{n-k} = -(-1)^k$ and:

$$\sum_{k=0}^{n}(-1)^k\ \zeta_{k,\rho}(s) \overset{(6.2.2)}{=} \sum_{k=0}^{n}(-1)^k\ \zeta_{n-k,\rho}(s) = \sum_{k=0}^{n}(-1)^{n-k}\ \zeta_{k,\rho}(s) = -\sum_{k=0}^{n}(-1)^k\ \zeta_{k,\rho}(s).$$

□

THEOREM 6.2.7 (2.3, [**65**]). Let X be a closed oriented manifold and ρ an orthogonal representation, not necessarily acyclic. If $\dim X = n$ is even, then $\sum_{k=0}^{n}(-1)^k k\ \zeta_{k,\rho}(s) = 0$ for each $s \in \mathbb{C}$. In particular, $\log T_X(\rho) = 0$ in even dimension.

By Proposition 6.2.6, we conclude:

COROLLARY 6.2.8. If $\dim X = n$ is even, then also $\sum_{k=0}^{n}(-1)^k\ \zeta_{k,\rho}(s) = 0$. Hence this is true $\forall n \in \mathbb{N}$.

REMARK 6.2.9 (§5.3.2, [**67**]). Since $\chi(X) = 0$ when X closed and odd-dimensional, the analytic torsion represents a complementary invariant for closed manifolds, able to distinguish between manifolds when the Euler characteristic cannot. In particular, given the relationship between χ and Index Theory, we can see that analytic torsion (and R-torsion as well) provides information when Index Theory fails to do so.

Although Ray and Singer in [**65**] could prove that analytic and R-torsion of closed oriented manifolds share important properties, they could only conjecture their equivalence. The conjecture was set for the affirmative by Cheeger and Müller, independently, around 1980:

THEOREM 6.2.10 (Theorem 10.22, [**55**]). Let X be a closed oriented manifold. Then $T_X(\rho) = \tau_X(\rho)$.

Finally, also in the light of Remark 6.1.28, we can also define R-torsion and analytic torsion for non-acyclic representations of X. However, in this case we have dependence on the Riemannian metric g of X. This is true also for manifolds with boundary, but we postpone the full statement of the theorem to the next section.

THEOREM 6.2.11 (Theorem 7.6, [**65**]). Let $u \in \mathbb{R} \mapsto g^X(u)$ be a smooth path of metrics[4]. If $C_r(X, \rho)$ is not acyclic, then:

$$\frac{d}{du} \log \tau_X(\rho) = \frac{d}{du} \log T_X(\rho) = \frac{1}{2} \sum_{k=0}^{n} (-1)^k \operatorname{tr} \alpha_{k|_{\ker \Delta_k}} \tag{6.2.3}$$

where $\alpha_k := *_k^{-1} \dot{*}_k : \Omega^k(X, E_\rho) \to \Omega^k(X, E_\rho)$.

We conclude this section by introducing another homotopy invariant which will appear in the next paragraphs.

DEFINITION 6.2.12 (§1, [**63**]). The integer $\chi'(X)$ defined as:

$$\chi'(X) := \sum_{k=0}^{n} (-1)^k k b_k, \qquad b_k = \dim H^k(X)$$

is called *secondary* (or *derived*) *Euler characteristic* of X.

REMARK 6.2.13. We remark that the above definition differs from the one in [**63**] by a sign and that the secondary Euler characteristic is the first of a sequence of homotopy invariants for (not necessarily closed) manifolds X called *higher Euler characteristics*:

$$\chi_j(X) := \sum_{k=0}^{n} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} b_k, \tag{6.2.4}$$

with clearly $\chi_0(X) = \chi(X)$ and $\chi_1(X) = -\chi'(X)$. It is interesting to note that (6.2.4) is not the only natural generalization of $\chi(X)$ and $\chi'(X)$. $\chi'(X)$ in particular has appeared recently in many fields; for instance, it is a term of the family analytic torsion studied in [**9**]. For more properties and references on higher Euler characteristics, we refer to [**63**].

PROPOSITION 6.2.14. Let X be a closed n-dimensional manifold. Then:

$$\chi'(X)(1 + (-1)^n) = n\chi(X).$$

[4]Which exists since the space of Riemannian metrics on X is convex, [**55**].

PROOF. By Poincaré duality:

$$\begin{aligned}\chi'(X) &= \sum_{k=0}^{n}(-1)^k k b_k = \sum_{k=0}^{n}(-1)^k k b_{n-k} = \sum_{k=0}^{n}(-1)^{n-k}(n-k)b_k \\ &= (-1)^n n \sum_{k=0}^{n}(-1)^k b_k + (-1)^{n-1}\sum_{k=0}^{n}(-1)^k k b_k \\ &= (-1)^{n-1}\chi'(X) + (-1)^n n\chi(X).\end{aligned}$$

□

COROLLARY 6.2.15. If n is even, then $\chi'(X) = \frac{n}{2}\chi(X)$.

REMARK 6.2.16. Corollary 6.2.15 shows that if $\chi(X)$ does not vanish, then $\chi'(X)$ does not really provide new information. However, $\chi'(X)$ provides information on X when n is odd and, in general, $\chi_j(X)$ is the first nontrivial homotopy invariant when $\chi_k(X)$ vanish for each $k < j$.

6.2.2. Exotic torsions of closed manifolds. We recall from Chapter 1 that a determinant is a homomorphism $\det_{\tau,\epsilon} = \epsilon \circ \tau \circ \log$, where $\epsilon : \mathcal{T} \to \mathcal{S}$ is an exponential map. If in addition ϵ has a left inverse, i.e. a (possibly different) logarithm map $\widetilde{\log} : \mathcal{S} \to \mathcal{T}$, then:

$$\widetilde{\log} \circ \det_{\tau,e} = \tau \circ \log .$$

With this in mind, we will re-write analytic and R-torsion in terms of the composition of logarithm and trace, i.e. as log-determinants.

PROPOSITION 6.2.17. Let X be a closed oriented manifold and ρ an orthogonal acyclic representation. Then:

$$\log \tau_X(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}k \operatorname{tr}\log \Delta_k^c.$$

PROOF. Since the combinatorial Laplacian Δ_k^c for an acyclic complex is a positive definite square matrix, by holomorphic functional calculus (see §1.3.1) we can define its logarithm as:

$$\log \Delta_k^c := \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda \ (\Delta_k^c - \lambda)^{-1} d\lambda, \tag{6.2.5}$$

where $\mathscr{C}$ is a closed loop around the spectrum of Δ_k^c. Thus, its eigenvalues are of the form $\log \lambda_i$, for $\lambda_i > 0$ an eigenvalues of Δ_k^c. Therefore, (6.1.4) implies that $\log\det\Delta_k^c = \operatorname{tr}\log\Delta_k^c$ and by Proposition 6.1.32 the statement follows.

□

REMARK 6.2.18. If Δ_k^c in not positive definite but only semi-definite, i.e. the complex is not acyclic, then the statement holds for the positive eigenvalues, i.e. for $\det(\Delta_k^c + \Pi_k)$, with Π_k the orthogonal projection onto $\ker \Delta_k^c$, by a standard regularization argument.

In a similar way, let us consider Δ_k, the restriction to k-forms of the Laplacian $\Delta : \Omega(X, E_\rho) \to \Omega(X, E_\rho)$ in §6.2.1. Since it is elliptic with non-negative real eigenvalues, it is admissible and for $\mathscr{C}$ a *Leurent loop* ((2.6) in [**22**], and §2 in [**78**]), i.e.:

$$\mathscr{C} := \{re^{i\pi} |\ \infty > r > r_0\} \cup \{r_0 e^{i\theta} |\ \pi \geq \theta \geq -\pi\} \cup \{re^{i\pi} |\ r_0 < r < \infty\}, \tag{6.2.6}$$

by holomorphic functional calculus ([**76**]),

$$\Delta_k^{-s} := \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} (\Delta_k - \lambda)^{-1} d\lambda, \tag{6.2.7}$$

is a holomorphic family for $\Re(s) > 0$, which is trace class if in particular $\Re(s) > \frac{n}{2}$, and defines a logarithm as $\log \Delta_k := -\frac{d}{ds}_{|s=0} \Delta_k^{-s}$, i.e.

$$\log \Delta_k = \lim_{s \searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda (\Delta_k - \lambda)^{-1} d\lambda = \frac{i}{2\pi} \int_{\mathscr{C}} \log \lambda\ (\Delta_k - \lambda)^{-1} d\lambda. \tag{6.2.8}$$

As shown in [**76**], the trace $\mathrm{Tr}(\Delta_k^{-s}) = \int_X k^{\Delta_k^{-s}}(x,x) dx$ extends meromorphically to $\mathbb{C}$ and such extension coincides to the spectral zeta function defined in Remark 6.2.1, i.e. $\zeta_{k,\rho}(s) = \mathrm{Tr}(\Delta_k^{-s})|^{\mathrm{mer}}$, which is holomorphic at $s = 0$; there, its derivative is:

$$\frac{d}{ds} \zeta_{k,\rho}(0) = \frac{d}{ds}_{|s=0} \mathrm{Tr}(\Delta_k^{-s})|^{\mathrm{mer}} = \lim_{s \searrow 0} \mathrm{Tr}(\frac{d}{ds} \Delta_k^{-s})|^{\mathrm{mer}} =: -\mathrm{TR}_\zeta(\log \Delta_k),$$

where TR_ζ is the *zeta quasi-trace*, the extension of the classical trace to $\Psi^{\mathbb{Z}}$ with respect to the complex power gauging. We refer to [**39**], and §1.5.6–§1.5.7 of [**75**], for a general description of the extension of the classical trace to elliptic pseudodifferential operator of any order via complex gauging.

Therefore, $\log \det_\zeta \Delta_k = -\zeta'_{k,\rho}(0) = \mathrm{TR}_\zeta(\log \Delta_k)$ and in conclusion we obtain:

PROPOSITION 6.2.19. Let X be a closed oriented manifold and ρ an orthogonal representation. Then:

$$\log T_X(\rho) = \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} k\ \mathrm{TR}_\zeta(\log \Delta_k).$$

REMARK 6.2.20 (Lemma 1.10.1, [**23**]). Since $\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}dt$ by Mellin transform and Δ_k is positive definite (in the case at hand), $\zeta_{k,\rho}(s)$ is equivalently defined as (the meromorphic extension of):

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\ \mathrm{Tr}(e^{-t\Delta_k})dt,$$

where $e^{t\Delta_k}$ is the heat semigroup associated to Δ_k and $\Gamma(s)$ is the *gamma function*. Moreover, if $A \in \Psi(X, \Lambda^k(X)\otimes E_\rho)$, a (generalized) zeta function is defined as

$$\zeta(A, \Delta_k, s) := \mathrm{Tr}(A\Delta_k^{-s})|^{\mathrm{mer}} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\ \mathrm{Tr}(Ae^{-t\Delta_k})dt|^{\mathrm{mer}}. \tag{6.2.9}$$

Consequently, we can consider a pre-existing torsion element, defined at the operator level:

DEFINITION 6.2.21. We define the *torsion logarithm* to be the operator:

$$\mathbb{T}_X(\rho) := \frac{1}{2}\bigoplus_{k=0}^{n}(-1)^{k+1}k\ \log\Delta_k \in \bigoplus_{k=0}^{n}\frac{\Psi^{0,1}_{\log}(X,\Lambda^k(X)\otimes E_\rho)}{[\Psi^{0,1}_{\log}(X,\Lambda^k(X)\otimes E_\rho), \Psi^{0,1}_{\log}(X,\Lambda^k(X)\otimes E_\rho)]},$$

where $\Psi^{0,1}_{\log}$ are the log-classical[5] ψdos of order 0 and log-degree 1.

In general, given an $(n+1)$-tuple $\beta = (\beta_0, \dots, \beta_n) \in \mathbb{R}^{n+1}$, a chain complex C and a log operator $\log : D_k \to \log D_k \in A_k$ for R-modules A_k, with D_k chain maps, we can define a *generalized* torsion logarithm as

$$\mathbb{T}^\beta_X(\rho) := \frac{1}{2}\bigoplus_{k=0}^{n}(-1)^{k+1}\beta_k\ \log\Delta_k \in \bigoplus_{k=0}^{n}\frac{A_k}{[A_k, A_k]}.$$

Then, given the regularized zeta-trace $\mathrm{TR}_\zeta : \Psi^{\mathbb{Z}}(X, \Lambda(X)\otimes E_\rho) \to \mathbb{C}$, Definition 6.2.21 yields $\log T_X(\rho) = \mathrm{TR}_\zeta \circ \mathbb{T}_X(\rho)$, i.e. a sum of log-determinants, and $T_X(\rho) = \exp(\mathrm{TR}_\zeta(\mathbb{T}_X(\rho)))$, i.e. a product of generalized determinants.

Therefore, we are going to investigate the effect of composing with other traces for $\Psi^{\leq 0}(X, \Lambda(X)\otimes E_\rho) := \bigcup_{m\leq 0}\Psi^m(X, \Lambda(X)\otimes E_\rho)$, as different trace evaluations of $\mathbb{T}_X(\rho)$ may generate different log-determinants of Δ and possibly different invariants for X. To this purpose, let us recall that the leading symbol σ^B of $B \in \Psi^m(X, E)$ is a globally defined section over the co-sphere bundle $S^*X \to X$. Then:

DEFINITION 6.2.22 (§1.5.8.3, [**75**]). The *leading symbol trace* is the linear map $\tau_0 : \Psi^{\leq 0}(X, E) \to C^\infty(S^*X)$ defined as $\tau_0(A)(x,\xi) = \mathrm{tr}\ \sigma^A(x,\xi)$.

REMARK 6.2.23 (§1.5.8.3, [**75**]). For $u \in \mathcal{D}'(S^*X)$ any distribution, then $\tau_{u,0} : \Psi^{\leq 0}(X, E) \to \mathbb{C}$, defined as $\tau_{u,0}(A) = u(\tau_0(A))$, is a scalar trace.

---

[5]For the generalization of classical ψdos to log-classical ψdos of order m and log-degree k we refer to §2.6.1.2 of [**75**].

We investigate the nature of the log-determinant arising with respect to this trace and define:

DEFINITION 6.2.24. The *(exotic) analytic leading symbol torsion* $T_X^{\text{lead},\beta,u}(\rho)$ associated to $u \in \mathcal{D}'(S^*X)$ is the character of $\mathbb{T}_X(\rho)$ with respect to a scalar leading symbol trace[6], i.e.

$$\log T_X^{\text{lead},\beta,u}(\rho) := \tau_{0,u}(\mathbb{T}_X^{\beta}(\rho)) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \ \tau_{0,u}\log\Delta_k, \ \forall\beta\in\mathbb{R}^{n+1}.$$

As it turns out, such torsion invariants vanish identically:

THEOREM 6.2.25. Let X be a closed oriented n-dimensional manifold. Then $\log T_X^{\text{lead},\beta,u}(\rho) = 0 \ \forall\beta\in\mathbb{R}^{n+1}$ and $u\in\mathcal{D}'(S^*X)$.

PROOF. Let $\sigma_0^{\log\Delta_k}(x,\xi)$ and $\sigma_2^{\Delta_k}(x,\xi)$ denote the principal symbols of $\log\Delta_k$ and Δ_k, respectively. Then by Proposition 2 of [**61**]:

$$\sigma_0^{\log\Delta_k}(x,\xi) = 2\log|\xi| I + \log\sigma_2^{\Delta_k}(x,\frac{\xi}{|\xi|}).$$

Since $\sigma_2^{\Delta_k}(x,\xi) = |\xi|^2 I$, we have that $\log\sigma_2^{\Delta_k}(x,\frac{\xi}{|\xi|}) = \log\left(\frac{|\xi|}{|\xi|}\right)^2 I = 0$; thus,

$$\tau_0(\Delta_k)(x,\xi) = 2\log|\xi| \ \mathrm{tr}I = 0 \quad \text{as} \quad (x,\xi)\in S^*X.$$

□

The leading symbol trace is only one of the two independent traces on $\Psi^{\leq 0}(X,E)$. In fact, every trace on $\Psi^{\leq 0}(X,E)$ is a linear combination of the leading symbol trace and the *residue trace* (§2.7.4, [**75**]):

DEFINITION 6.2.26 (§1.5.4, [**75**]). The *residue trace* is a continuous functional $\mathrm{res}: \Psi^{\mathbb{Z}}(X,\Lambda(X)\otimes E_\rho)\to\mathbb{C}$ defined as

$$\mathrm{res}(A) := \int_X \left(\int_{|\xi|=1} \mathrm{tr}\ \sigma^A_{-\dim X}(x,\xi)\ d_\xi S\right) dx$$

It is the unique trace on classical pseudodifferential operators $\Psi^{\mathbb{Z}}(X,E)$ and is (roughly) complementary to TR_ζ.

Hence, in this context, the residue trace becomes the unique trace at hand and we can use it to define:

[6] In practice, it's extension to $\Psi^{0,1}_{\log}$.

DEFINITION 6.2.27. The *(exotic) analytic residue torsion* $T_X^{\mathrm{res},\beta}(\rho)$ is the character of $\mathbb{T}_X(\rho)$ with respect to the residue trace, i.e.

$$\log T_X^{\mathrm{res},\beta}(\rho) := \mathrm{res}(\mathbb{T}_X^{\beta}(\rho)) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \ \mathrm{res}\log\Delta_k, \ \forall\beta\in\mathbb{R}^{n+1}.$$

It is, as we shall see, an invariant for X of a complementary behaviour with respect to the classical analytic torsion.

THEOREM 6.2.28. Let X be a closed oriented n-dimensional manifold and $\rho : \pi_1(X) \to O(N)$ an orthogonal representation (not necessarily acyclic). Then:

(i) if n is odd, $\log T_X^{\mathrm{res},\beta}(\rho) = 0 \ \forall\beta\in\mathbb{R}^{n+1}$;

(ii) if n is even, $\log T_X^{\mathrm{res},\beta}(\rho)$ is a smooth invariant if β equals:

$$\underline{1} := (1,\ldots,1) \quad \text{or} \quad \underline{\omega} := (0,1,\ldots,n). \tag{6.2.10}$$

The corresponding residue analytic torsions are equal, respectively, to the Euler characteristic χ and the *derived* Euler characteristics χ' (Definition 6.2.12):

$$\log T_X^{\mathrm{res}}(\rho) := \log T_X^{\mathrm{res},\underline{1}}(\rho) = \chi(X,E_\rho) = \chi(X)\mathrm{rk}(E_\rho) \qquad \text{and}$$

$$\log T_X^{\mathrm{res}}(\rho)' := \log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \chi'(X,E_\rho) = \chi'(X)\mathrm{rk}(E_\rho). \tag{6.2.11}$$

Finally, for a smooth path of metrics $u\in\mathbb{R}\to g_X(u)$ we have:

$$\frac{d}{du}\log T_X^{\mathrm{res}}(\rho)' = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\ \underbrace{\mathrm{res}(\alpha_k)}_{=0}, \tag{6.2.12}$$

i.e. it is a smooth invariant (and even if it vanishes, it has the same form of (6.2.3)).

PROOF. (i) Let n be odd. Since differential operators and their inverses are odd-class[7] by Lemma 7.1, [**39**], so are $\Delta_k - \lambda I$ and $(\Delta_k-\lambda I)^{-1}$, with $\lambda\notin\mathrm{spec}(\Delta_k)$. Moreover, as the symbol of $\log\Delta_k\in\Psi^0(X,\Lambda(X)\otimes E_\rho)$ has asymptotic expansion:

$$\sigma^{\log\Delta_k}(x,\xi)\sim\sum_{j\geq 0}\sigma_j^{\log\Delta_k}(x,\xi) = 2\log|\xi| + \sum_{j\geq 2}\frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda\ \sigma_{-j}^{(\Delta_k-\lambda)^{-1}}(x,\xi)\ d\lambda,$$

we have $\sigma_{-n}^{\log_\theta\Delta_k}(x,\xi) = \frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda\ \sigma_{-n}^{(\Delta_k-\lambda)^{-1}}(x,\xi)\ d\lambda$, which is odd in ξ because n is odd, i.e.:

$$\sigma_{-n}^{\log_\theta\Delta_k}(x,\xi) = \frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda\ \sigma_{-n}^{(\Delta_k-\lambda)^{-1}}(x,\xi)\ d\lambda =$$

[7] See §0.2.

$$= \frac{(-1)^{-n} i}{2\pi} \int_{\mathscr{C}} \log\lambda \; \sigma_{-n}^{(\Delta_k-\lambda)^{-1}}(x,-\xi)\; d\lambda = -\sigma_{-n}^{\log_\theta \Delta_k}(x,-\xi).$$

We remark that, generally, the various homogeneous terms $\sigma_j^{\log \Delta_k}(x,\xi)$ do not define a global density on X, while $\sigma_{-n}^{\log_\theta \Delta_k}(x,\xi)$ does, as observed by Okikiolu in [**60**]. Therefore[8], $\int_{|\xi|=1} \mathrm{tr}\; \sigma_{-n}^{\log_\theta \Delta_k}(x,\xi)\; d_\xi S = 0$ and $\mathrm{res} \log\Delta_k = 0$, which clearly yields $\log T_X^{\mathrm{res},\beta}(\rho) = 0$ for each $\beta \in \mathbb{R}^{n+1}$.

(ii) Let now n be even and $u \in \mathbb{R} \mapsto g^X(u)$ be a smooth path of metrics. Since $\log T_X^{\mathrm{res},\beta}(\rho)$ is a smooth invariant if and only if it is independent of the Riemannian metric g_X, i.e. $\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho) = 0$, we need to compute $\frac{d}{du}\mathrm{res}\log\Delta_k$. We know that the Hodge operator depends smoothly on the metric g_X, so $*_k = *_k(u)$ is a smooth family for each k and $\Delta_k = \Delta_k(u)$ is an admissible smooth family of constant order.

Since $*_k^{-1} = (-1)^k *_{n-k}$ and $\frac{d}{du}(*_k^{-1} *_k) = \frac{d}{du}(id) = 0$,

$$0 = \frac{d}{du}(*_k^{-1}) *_k + *_k^{-1} \frac{d}{du}(*_k) = \dot{*}_{n-k} \; *_{n-k}^{-1} + *_k^{-1} \; \dot{*}_k,$$

and if we set $\alpha_k := *_k^{-1} \; \dot{*}_k = - \; \dot{*}_{n-k} \; *_{n-k}^{-1} : \Lambda^k(X,E_\rho) \to \Lambda^k(X,E_\rho)$, we have:

$$\text{(6.2.13)} \quad \dot{\Delta}_k := \frac{d}{du}\Delta_k = \frac{d}{du}(\delta_k d_k + d_{k-1}\delta_{k-1})$$
$$= -\alpha_k\delta_k d_k + \delta_k \alpha_{k+1} d_k - d_{k-1}\alpha_{k-1}\delta_{k-1} + d_{k-1}\delta_{k-1}\alpha_k,$$

as in the proof of Theorem 2.1, [**65**]. Notice the difference in sign due to our definition of Laplacian.

As all Δ_k have spectrum on the non-negative real axis, we can consider a Laurent loop $\mathscr{C}$ independent of u, thus defininig a differentiable family $\log\Delta_k$, of constant order 0. Hence, by Proposition 7 of [**61**], we have:

$$\frac{d}{du}\mathrm{res}\log\Delta_k = \mathrm{res}\left(\frac{d}{du}\log\Delta_k\right).$$

Let Π_k be the orthogonal projection onto $\ker\Delta_k \cong H^k(X)$; as $H^k(X)$ is an homotopy invariant, Π_k is a finite rank operator that does not dependent on the metric and $\Delta_k + \Pi_k$ are a differentiable family of invertible operators of constant order. This yields $\frac{d}{du}(\Delta_k + \Pi_k) = \dot{\Delta}_k$.

---

[8] Clearly, if f is an odd function on $\mathbb{R}^N$, N odd, and S^{N-1} is the $(N-1)$-dimensional sphere

$$\int_{S^{N-1}} f(x) d_x S = -\int_{S^{N-1}} f(-x) d_x S = -\int_{S^{N-1}} f(x) d_x S.$$

Since $\mathrm{spec}(\Delta_k+\Pi_k)=\mathrm{spec}(\Delta_k)\setminus\{0\}$, $\log(\Delta_k+\Pi_k)$ and $\log\Delta_k$ can be defined for the same contour $\mathscr{C}$ and

$$\begin{aligned}\log\Delta_k-\log(\Delta_k+\Pi_k) &= \frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda(\Delta_k-\lambda)^{-1}d\lambda-\frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda(\Delta_k+\Pi_k-\lambda)^{-1}d\lambda\\ &= \frac{i}{2\pi}\int_{\mathscr{C}}\log\lambda[(\Delta_k-\lambda)^{-1}-(\Delta_k+\Pi_k-\lambda)^{-1}]\,d\lambda.\end{aligned}$$

Since $(\Delta_k-\lambda)^{-1}-(\Delta_k+\Pi_k-\lambda)^{-1}=(\Delta_k+\Pi_k-\lambda)^{-1}\Pi_k(\Delta_k-\lambda)^{-1}$, we can conclude that

$$\log\Delta_k-\log(\Delta_k+\Pi_k)\in\Psi^{-\infty}(X,\Lambda(X)\otimes E_\rho).$$

In particular,

$$\frac{d}{du}\log\Delta_k-\frac{d}{du}\log(\Delta_k+\Pi_k)\in\Psi^{-\infty}(X,\Lambda(X)\otimes E_\rho).$$

Since the residue trace vanishes on $\Psi^{-\infty}(X,\Lambda(X)\otimes E_\rho)$, we have

$$\mathrm{res}\left(\frac{d}{du}\log\Delta_k\right)=\mathrm{res}\left(\frac{d}{du}\log(\Delta_k+\Pi_k)\right).$$

By Lemma 1, [**61**], we also have:

$$\frac{d}{du}\log(\Delta_k+\Pi_k)=\dot{\Delta}_k(\Delta_k+\Pi_k)^{-1}+S\in\Psi^0(X,\Lambda T^*X\otimes E_\rho),$$

where S is a sum of commutators. Hence, in conclusion:

$$\frac{d}{du}\mathrm{res}\log\Delta_k=\mathrm{res}\left(\frac{d}{du}\log(\Delta_k+\Pi_k)\right)=\mathrm{res}\left(\dot{\Delta}_k(\Delta_k+\Pi_k)^{-1}\right)=\mathrm{res}\left((\Delta_k+\Pi_k)^{-1}\dot{\Delta}_k\right).$$

We remark that $P_k:=(\Delta_k+\Pi_k)^{-1}$ is a parametrix for Δ_k, since:

$$I=P_k(\Delta_k+\Pi_k)=P_k\Delta_k+P_k\Pi_k,\qquad\text{where}\qquad P_k\Pi_k\in\Psi^{-\infty}(X,\Lambda(X)\otimes E_\rho).$$

For the sake of notation, we will only write $\Psi^{-\infty}$ from now on. By (6.2.13) and the linearity of res, we write:

$$\begin{aligned}\mathrm{res}(P_k\dot{\Delta}_k) \quad = \quad & \underbrace{-\mathrm{res}(P_k\alpha_k\delta_kd_k)}_{1}+\underbrace{\mathrm{res}(P_k\delta_k\alpha_{k+1}d_k)}_{2}\\ & \underbrace{-\mathrm{res}(P_kd_{k-1}\alpha_{k-1}\delta_{k-1})}_{3}+\underbrace{\mathrm{res}(P_kd_{k-1}\delta_{k-1}\alpha_k)}_{4}\end{aligned}$$

The following identities for the Laplacian:

$$d_k\Delta_k=\Delta_{k+1}d_k\qquad\delta_{k-1}\Delta_k=\Delta_{k-1}\delta_{k-1}\tag{6.2.14}$$

hold also for the parametrix P_k. In fact, since $\Delta_kP_k-I\in\Psi^{-\infty}$, we have that both $d_k\Delta_kP_k-d_k$ and $\Delta_{k+1}P_{k+1}d_k-d_k$ are smoothing. So after subtracting these terms, by (6.2.14) we obtain $\Delta_{k+1}(d_kP_k-P_{k+1}d_k)\in\Psi^{-\infty}$. Hence $d_kP_k-P_{k+1}d_k\in\Psi^{-\infty}$

and $\delta_{k-1}P_k - P_{k-1}\delta_{k-1} \in \Psi^{-\infty}$ can be proved in the same way. These two identities can be used to rearrange (1) and (3):

$$\begin{aligned}
(1) &= -\mathrm{res}(P_k\alpha_k\delta_k d_k) = -\mathrm{res}(\delta_k d_k P_k\alpha_k) = -\mathrm{res}(\delta_k P_{k+1} d_k\alpha_k) \\
&= -\mathrm{res}(P_k\delta_k d_k\alpha_k) \\
(3) &= -\mathrm{res}(P_k d_{k-1}\alpha_{k-1}\delta_{k-1}) = -\mathrm{res}(\delta_{k-1}P_k d_{k-1}\alpha_{k-1}) \\
&= -\mathrm{res}(P_{k-1}\delta_{k-1}d_{k-1}\alpha_{k-1})
\end{aligned}$$

On the other hand, since $\alpha_k - P_k\Delta_k\alpha_k \in \Psi^{-\infty}$, we can decompose:

$$\alpha_k - P_k\delta_k d_k\alpha_k - P_k d_{k-1}\delta_{k-1}\alpha_k \in \Psi^{-\infty}$$

and use this to rearrange (2) and (4):

$$\begin{aligned}
(2) &= \mathrm{res}(P_k\delta_k\alpha_{k+1}d_k) = \mathrm{res}(d_k P_k\delta_k\alpha_{k+1}) = \mathrm{res}(P_{k+1}d_k\delta_k\alpha_{k+1}) \\
&= \mathrm{res}(\alpha_{k+1}) - \mathrm{res}(P_{k+1}\delta_{k+1}d_{k+1}\alpha_{k+1}) \\
(4) &= \mathrm{res}(P_k d_{k-1}\delta_{k-1}\alpha_k) = \mathrm{res}(\alpha_k) - \mathrm{res}(P_k\delta_k d_k\alpha_k)
\end{aligned}$$

Now, for $\gamma_k := \mathrm{res}(P_k\delta_k d_k\alpha_k)$ we can write:

$$(1) = -\gamma_k, \quad (2) = \mathrm{res}(\alpha_{k+1}) - \gamma_{k+1}, \quad (3) = -\gamma_{k-1}, \quad \text{and } (4) = \mathrm{res}(\alpha_k) - \gamma_k,$$

thus obtaining:

$$\mathrm{res}(P_k\dot{\Delta}_k) = \mathrm{res}(\alpha_k) + \mathrm{res}(\alpha_{k+1}) - \gamma_{k+1} - 2\gamma_k - \gamma_{k-1}.$$

Note that $\mathrm{res}(\alpha_k) = 0$, since $\alpha_k \in \mathrm{End}(\Lambda^k(M, E_\rho))$, and $\gamma_0 = \mathrm{res}(\alpha_0)$. Then:

$$\begin{aligned}
&2\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho) = \\
&= \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \frac{d}{du}\mathrm{res}\log\Delta_k = \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \mathrm{res}(P_k\dot{\Delta}_k) \\
&= \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \mathrm{res}(\alpha_k) + \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \mathrm{res}(\alpha_{k+1}) \\
&- 2\sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \gamma_k - \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \gamma_{k+1} - \sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \gamma_{k-1} \\
&= -\beta_0\ \mathrm{res}(\alpha_0) + \sum_{k=1}^n (-1)^{k+1}(\beta_k - \beta_{k-1})\ \mathrm{res}(\alpha_k) \\
&- 2\sum_{k=0}^n (-1)^{k+1}\ \beta_k\ \gamma_k + \sum_{k=1}^n (-1)^{k+1}\ \beta_{k-1}\ \gamma_k + \sum_{k=0}^{n-1} (-1)^{k+1}\ \beta_{k+1}\ \gamma_k \\
&= \sum_{k=0}^n (-1)^{k+1}(\beta_k - \beta_{k-1})\ \underbrace{\mathrm{res}(\alpha_k)}_{=0} + \sum_{k=0}^n (-1)^{k+1}\ (\beta_{k+1} - 2\beta_k + \beta_{k-1})\ \gamma_k.
\end{aligned}$$

where we can require $\beta_{-1} = 0$ since $\gamma_n = 0$.

At this point, we seek those $\beta \in \mathbb{R}^{n+1}$ such that $\beta_{k+1} - 2\beta_k + \beta_{k-1} = 0$, i.e a solution of the recurrence equation $\beta_{k+1} = 2\beta_k - \beta_{k-1}$. As the characteristic polynomial is $x^2 - 2x + 1$, a general solution must be a linear combination of the two independent solutions $\beta_k = 1$ and $\beta_k = k$, for each $k = 0, \dots, n$, i.e:

$$\underline{1} = (1, \dots, 1) \quad \text{or} \quad \underline{\omega} = (0, 1, \dots, n).$$

Hence, for $\beta = \underline{1}$ or $\beta = \underline{\omega}$, we can conclude $\frac{d}{du} \log T_X^{\mathrm{res},\beta}(\rho) = 0$ and $\log T_X^{\mathrm{res},\beta}(\rho)$ does not depend on the Riemannian metric.

Now, from Theorem 1.8 of [**74**], we have:

$$-\frac{1}{2}\mathrm{res}\log\Delta_k = \zeta_{k,\rho}(0) + \dim\ker(\Delta_k), \tag{6.2.15}$$

which allows us to write:

$$\log T_X^{\mathrm{res},\beta}(\rho) = \sum_{k=0}^{n} (-1)^k \beta_k \zeta_{k,\rho}(0) + \sum_{k=0}^{n} (-1)^k \beta_k \dim\ker(\Delta_k).$$

Thus, if $\beta = \underline{1}$,

$$\log T_X^{\mathrm{res},\underline{1}}(\rho) = \sum_{k=0}^{n} (-1)^k \zeta_{k,\rho}(0) + \sum_{k=0}^{n} (-1)^k \dim\ker(\Delta_k) = 0 + \chi(X, E_\rho)$$

by Corollary 6.2.8, while if $\beta = \underline{\omega}$, since n is even,

$$\log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \sum_{k=0}^{n} (-1)^k k\ \zeta_{k,\rho}(0) + \sum_{k=0}^{n} (-1)^k k\ \dim\ker(\Delta_k) = 0 + \chi'(X, E_\rho)$$

by Theorem 6.2.7.

□

Corollary 6.2.29. If $\beta = \underline{1}$ or $\beta = \underline{\omega}$ then $\log T_X^{\mathrm{res},\beta}(\rho)$ is a smooth invariant. In fact,

$$\log T_X^{\mathrm{res},\underline{1}}(\rho) = \chi(X, E_\rho) \qquad \text{and} \qquad \log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \frac{n}{2}\chi(X, E_\rho).$$

Remark 6.2.30. The res-log of a generalized Laplacian is linked to the index of the associated Dirac operator. In fact, by (6.2.15), we can write

$$\mathrm{ind}(d+\delta)^+ = \frac{1}{2}\left(\mathrm{res}\log(d+\delta)^+(d+\delta)^- - \mathrm{res}\log(d+\delta)^-(d+\delta)^+\right).$$

This can be accounted for the fact that the behaviour of the residue torsion is complementary to the one of the analytic torsion, as Theorem 6.2.28 showed.

REMARK 6.2.31. Equivalently, since $\zeta_k(0) = -\dim H^k(X)$ when $\dim X$ is odd (Theorem 7.6, [**55**]), part (i) follows directly from Scott's formula (6.2.15), which could also be used to prove part (ii), together with an approach similar to the one of Theorem 2.1 in [**65**]. Indeed, we can express the residue torsion as:

$$\log T_X^{\mathrm{res},\beta}(\rho) = \sum_{k=0}^{n} (-1)^k \beta_k \ \zeta_{k,\rho}(0) + \sum_{k=0}^{n} (-1)^k \beta_k \ \dim\ker(\Delta_k)$$

and calculate $\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho)$ in this case. As $\ker(\Delta_k) \cong H^k(X)$, it is independent of the metric and we have:

$$\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho) = \frac{d}{du}\sum_{k=0}^{n} (-1)^k \beta_k \ \zeta_{k,\rho}(0).$$

We can therefore evaluate for $s = 0$ the derivative with respect to u of the meromorphic extension of

$$f(u,s) := \sum_{k=0}^{n} (-1)^k \beta_k \ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta_k(u)}) dt,$$

which is well-known to be analytic at $s = 0$. Then the statement will follow from, $\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho) = \frac{\partial}{\partial u} f(u,0)$.

By the proof of Theorem 2.1, [**65**], we can differentiate under the integral sign, thus obtaining:

$$\begin{aligned}
\frac{\partial}{\partial u} f(u,s) &= \sum_{k=0}^{n} (-1)^k \beta_k \ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\partial}{\partial u}\mathrm{Tr}(e^{-t\Delta_k(u)}) dt \\
&= \sum_{k=0}^{n} (-1)^{k+1} \beta_k \ \frac{1}{\Gamma(s)} \int_0^\infty t^{s} \frac{\partial}{\partial u}\mathrm{Tr}\left(e^{-t\Delta_k(u)}\dot{\Delta}_k(u)\right) dt,
\end{aligned} \tag{6.2.16}$$

as $\frac{\partial}{\partial u}\mathrm{Tr}\left(e^{-t\Delta_k}\right) = -t \ \mathrm{Tr}\left(e^{-t\Delta_k}\dot{\Delta}_k\right)$. By (6.2.13), (6.2.14), and the traciality of Tr, we can write:

$$\mathrm{Tr}\left(e^{-t\Delta_k}\dot{\Delta}_k\right) = -\mathrm{Tr}\left(e^{-t\Delta_k}\delta d\alpha\right) + \mathrm{Tr}\left(e^{-t\Delta_{k+1}} d\delta\alpha\right) - \mathrm{Tr}\left(e^{-t\Delta_{k-1}}\delta d\alpha\right) + \mathrm{Tr}\left(e^{-t\Delta_k} d\delta\alpha\right).$$

If we set $\varphi_k := \mathrm{Tr}\left(e^{-t\Delta_k} d\delta\alpha\right)$ and $\theta_k := \mathrm{Tr}\left(e^{-t\Delta_k}\delta d\alpha\right)$, then

$$\mathrm{Tr}\left(e^{-t\Delta_k}\dot{\Delta}_k\right) = \varphi_{k+1} - \theta_k + \varphi_k - \theta_{k-1}$$

and we can rewrite (6.2.16) as:

$$\frac{\partial}{\partial u} f(u,s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \sum_{k=0}^{n} (-1)^{k+1} \beta_k \left(\varphi_{k+1} - \theta_k + \varphi_k - \theta_{k-1}\right) dt.$$

By standard manipulations, we have:

$$\sum_{k=0}^{n} (-1)^{k+1} \beta_k \left(\varphi_{k+1} - \theta_k + \varphi_k - \theta_{k-1}\right) = \sum_{k=1}^{n-1} (-1)^{k+1} \left[(\beta_k - \beta_{k-1})\varphi_k + (\beta_{k+1} - \beta_k)\theta_k\right]$$

$$+ (\beta_0 - \beta_1)\theta_0 + (-1)^n(\beta_{n-1} - \beta_n)\varphi_n =: \star$$

since $\varphi_0 = \theta_n = 0$. If we also set $\psi_k := \mathrm{Tr}\left(e^{-t\Delta_k}\Delta_k\alpha\right) = \varphi_k + \theta_k$, i.e. $\theta_k = \psi_k - \varphi_k$, we can write:

$$\begin{aligned}
\star &= \sum_{k=1}^{n}(-1)^{k+1}(2\beta_k - \beta_{k-1} - \beta_{k+1})\varphi_k + \sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)\mathrm{Tr}\left(e^{-t\Delta_k}\Delta_k\alpha\right) \\
&= \sum_{k=1}^{n}(-1)^{k+1}(2\beta_k - \beta_{k-1} - \beta_{k+1})\varphi_k + \sum_{k=0}^{n}(-1)^{k}(\beta_{k+1} - \beta_k)\frac{d}{dt}\mathrm{Tr}\left(e^{-t\Delta_k}\alpha\right).
\end{aligned}$$

Hence, (6.2.16) becomes

$$\begin{aligned}
\frac{\partial}{\partial u}f(u,s) = &\underbrace{\sum_{k=1}^{n}(-1)^{k}(\beta_{k+1} - 2\beta_k + \beta_{k-1})\frac{1}{\Gamma(s)}\int_0^\infty t^s\varphi_k dt}_{1} \\
&+ \underbrace{\sum_{k=0}^{n}(-1)^{k}(\beta_{k+1} - \beta_k)\frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{d}{dt}\mathrm{Tr}\left(e^{-t\Delta_k}\alpha\right)dt}_{2}.
\end{aligned}$$

On the one hand, via integration by parts, 2 becomes

$$\sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}\left(e^{-t\Delta_k}\alpha\right)dt =: \sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)s\zeta(\alpha,\Delta_k,s).$$

Since $\mathrm{res}(\alpha) = 0$, $\zeta(\alpha,\Delta_k,s)$ is regular at $s = 0$ and $s\zeta(\alpha,\Delta_k,s)$ vanishes there; thus $2 = 0$.

On the other hand,

$$\begin{aligned}
\frac{1}{\Gamma(s)}\int_0^\infty t^s\varphi_k dt = \frac{1}{\Gamma(s)}\int_0^\infty t^s\mathrm{Tr}\left(e^{-t\Delta_k}d\delta\alpha\right)dt &= \frac{\Gamma(s+1)}{\Gamma(s)}\zeta(\Delta_k^{-1}d\delta\alpha,\Delta_k,s) \\
&= s\zeta(\Delta_k^{-1}d\delta\alpha,\Delta_k,s), \qquad \Gamma(s+1) = s\Gamma(s),
\end{aligned}$$

is holomorphic at $s = 0$ and $\lim_{s\to 0} s\zeta(\Delta_k^{-1}d\delta\alpha,\Delta_k,s) = \frac{1}{2}\mathrm{res}(\Delta_k^{-1}d\delta\alpha)$. Therefore, for $s = 0$, 1 vanishes if $\beta_{k+1} - 2\beta_k + \beta_{k-1} = 0$, which has solutions (6.2.10) as in the proof of Theorem 6.2.28. Thus, $\frac{d}{du}\log T_X^{\mathrm{res},\beta}(\rho) = \frac{\partial}{\partial u}f(u,0) = 0$ for (6.2.10).

Finally, (6.2.12) can be retrieved from 2 in the following way. In fact, $\lim_{s\to 0} s\zeta(\alpha_k,\Delta_k,s) = \frac{1}{2}\mathrm{res}(\alpha_k)$ and for $\beta = \underline{\omega}$ we have:

$$\frac{d}{du}\log T_X^{\mathrm{res},\underline{\omega}}(\rho) = \frac{\partial}{\partial u}f(u,0) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\mathrm{res}(\alpha_k).$$

. □

An application of the same technique yields a motivation for the weights in the definition of the analytic torsion. In fact, by Definition 6.2.21, we could study a more general analytic torsion,

$$\log T_X^{\beta}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \log \det{}_{\zeta} \Delta_k, \qquad \beta \in \mathbb{R}^{n+1},$$

and find the weights β that correspond to a smooth invariant of X, in analogy with Theorem 6.2.28. In fact, (6.2.15) can be accounted for the underlying similarity of the results for analytic and residue analytic torsions.

THEOREM 6.2.32. Let X be a closed oriented n-dimensional manifold, with n odd and $\rho : \pi_1(X) \to O(N)$ an acyclic orthogonal representation. Then the generalized analytic torsion $\log T_X^{\beta}(\rho)$ is a smooth invariant if β equals:

$$\underline{1} = (1, \ldots, 1) \quad \text{or} \quad \underline{\omega} = (0, 1, \ldots, n). \tag{6.2.17}$$

If $\beta = \underline{1}$ we have that the $\log T_X^{\underline{1}}(\rho)$ vanishes identically.

PROOF. The proof is a generalization of the proof of Theorem 2.1, [**65**], and Remark 6.2.31. Set

$$f(u,s) := \frac{1}{2}\sum_{k=0}^{n}(-1)^k\beta_k \int_0^{\infty} t^{s-1}\mathrm{Tr}\left(e^{-t\Delta_k}\right) dt, \qquad \Re(s) \gg 0.$$

Then $f(u,0) = \log T_X^{\beta}(\rho)$ and for $\Re(s)$ large:

$$\frac{\partial}{\partial u} f(u,s) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \int_0^{\infty} t^{s}\mathrm{Tr}\left(e^{-t\Delta_k}\dot{\Delta}_k\right) dt. \tag{6.2.18}$$

If we set $\varphi_k := \mathrm{Tr}\left(e^{-t\Delta_k} d\delta\alpha\right)$ and $\theta_k := \mathrm{Tr}\left(e^{-t\Delta_k}\delta d\alpha\right)$ as in Remark 6.2.31, then by the same manipulation we obtain

$$\begin{aligned}\frac{\partial}{\partial u} f(u,s) = & \underbrace{\frac{1}{2}\int_0^{\infty} t^s \sum_{k=1}^{n}(-1)^k(\beta_{k+1} - 2\beta_k + \beta_{k-1})\varphi_k dt}_{1} \\ & \underbrace{+ \frac{1}{2}\sum_{k=0}^{n}(-1)^k(\beta_{k+1} - \beta_k)\int_0^{\infty} t^s \frac{d}{dt}\mathrm{Tr}\left(e^{-t\Delta_k}\alpha\right) dt}_{2}.\end{aligned}$$

By integration by parts, 2 becomes

$$\frac{s}{2}\sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)\underbrace{\int_0^{\infty} t^{s-1}\mathrm{Tr}\left(e^{-t\Delta_k}\alpha\right) dt}_{=:g(u,s)}.$$

and since $g(u,s)$ has no pole at $s=0$ (Theorem 2.1, [**65**]), 2=0 at $s=0$. On the other hand,

$$\int_0^\infty t^s \mathrm{Tr}\left(e^{-t\Delta_k} d\delta\alpha\right) dt = \Gamma(s+1)\mathrm{Tr}(\Delta_k^{-s}\Delta_k^{-1}d\delta\alpha)|^{\mathrm{mer}} = \Gamma(s+1)\zeta(\Delta_k^{-1}d\delta\alpha, \Delta_k, s)$$

is holomorphic at $s=0$, since $\mathrm{res}(\Delta_k^{-1}d\delta\alpha)=0$ as $\Delta_k^{-1}d\delta\alpha$ is odd class (as in the proof of (i) of Theorem 6.2.28). Therefore, for $s=0$,

$$1 = \frac{1}{2}\sum_{k=1}^{n}(-1)^k(\beta_{k+1}-2\beta_k+\beta_{k-1})\zeta(\Delta_k^{-1}d\delta\alpha, \Delta_k, 0) = 0$$

if $\beta_{k+1}-2\beta_k+\beta_{k-1}=0$, which has solutions (6.2.10) as in the proof of Theorem 6.2.28. Thus, $\frac{d}{du}\log T_X^{\beta}(\rho) = \frac{\partial}{\partial u}f(u,0) = 0$ when (6.2.10) hold.

The fact that $\log T_X^{\underline{1}}(\rho) = 0$ follows from Proposition 6.2.6.

□

COROLLARY 6.2.33. Let X be closed and let $\mathbb{T}_X^{\beta}(\rho) = \frac{1}{2}\bigoplus_{k=0}^{n}(-1)^{k+1}\beta_k \,\log\Delta_k$ be its logarithmic torsion, for a choice of orthogonal representation ρ. If n is even, non-trivial torsion invariants are the e residue torsions for $\beta=\underline{1}$ or $\beta=\underline{\omega}$ and coincide with the classical or derived Euler characteristics:

$$\log T_X^{\mathrm{res}}(\rho)' = \mathrm{res}(\mathbb{T}_M^{\underline{\omega}}(\rho)) = \chi'(X,E_\rho) = \frac{n}{2}\chi(X,E_\rho) = \frac{n}{2}\log T_X^{\mathrm{res},\underline{1}}(\rho),$$

while if n is odd, a non-trivial torsion invariant is the analytic torsion for $\beta=\underline{\omega}$ and coincides with the R-torsion:

$$\mathrm{TR}_\zeta(\mathbb{T}_M^{\underline{\omega}}(\rho)) = \log T_X(\rho) = \log\tau_X(\rho).$$

COROLLARY 6.2.34. The class of the logarithmic torsion $\mathbb{T}_X^{\beta}(\rho)\in\Psi^{\mathbb{Z}}/[\Psi^{\mathbb{Z}},\Psi^{\mathbb{Z}}]$ for $\beta=\underline{1}$ or $\beta=\underline{\omega}$ does not depend on the metric and therefore is a smooth invariant of X.

PROOF. Theorem 6.2.28 shows that the residue torsion is a smooth invariant if $\beta=\underline{1}$ or $\beta=\underline{\omega}$. Since res is the unique trace for $\Psi^{\mathbb{Z}} := \Psi^{\mathbb{Z}}(X,\Lambda(X)\otimes E_\rho)$ (§1.5.4, [**75**]), it pushes down to an isomorphism $\widetilde{\mathrm{res}} : \Psi^{\mathbb{Z}}/[\Psi^{\mathbb{Z}},\Psi^{\mathbb{Z}}]\cong\mathbb{C}$ by Lemma 1.2.4. Hence,

$$\widetilde{\mathrm{res}}\left(\frac{d}{du}\mathbb{T}_M^{\beta}(\rho)\right) = \frac{d}{du}\widetilde{\mathrm{res}}\left(\mathbb{T}_M^{\beta}(\rho)\right) = \frac{d}{du}\log T_M^{\mathrm{res},\beta}(\rho) = 0 \Longrightarrow \frac{d}{du}\mathbb{T}_M^{\beta}(\rho) = 0,$$

i.e. $\frac{d}{du}\mathbb{T}_M^{\beta}(\rho)\in[\Psi^{\mathbb{Z}},\Psi^{\mathbb{Z}}]$.

□

6.3. Torsion as a LogTQFT

Theorem 6.2.28 can be used to define a LogTQFT. Let us consider

$$F_{\mathbb{Z}} : \mathbf{Cob}_n^* \to \mathbb{C}\text{-}\mathbf{Alg}, \qquad M \mapsto F_{\mathbb{Z}}(M) := \Psi^{\mathbb{Z}}(M, \Lambda(M)).$$

Then $F_{\mathbb{Z}}$ is a strict pretracial monoidal product representation (see (2.24) of §2.1, [**72**], for the proof). Thus, for $\overline{X} \in \mathrm{mor}(M_0, M_1)$, $\partial X = Y_0^- \sqcup Y_1$, we can define a simplicial map $\log : \mathcal{N}\mathbf{Cob}_n \to F_{\mathbb{Z},\Pi}(\mathbf{Cob}_n^*)$ as

(6.3.1)

$$\log_{M_0 \sqcup M_1} \overline{X} := \pi_{M_0 \sqcup M_1} \circ \kappa_{\sharp} \left(\frac{1}{2} \bigoplus_{k=0}^{n} (-1)^k \beta_k \log \Delta_{k,Y_0} \oplus \frac{1}{2} \bigoplus_{k=0}^{n} (-1)^{k+1} \beta_k \log \Delta_{k,Y_1} \right),$$

with $\pi_{M_0 \sqcup M_1} \circ \kappa_{\sharp} : F(Y_0 \sqcup Y_1) \to F(M_0 \sqcup M_1)/[F(M_0 \sqcup M_1), F(M_0 \sqcup M_1)]$ as usual. Then, with respect to the residue trace, we obtain as character:

$$\begin{aligned} \mathrm{res}\left(\log^{\beta}_{M_0 \sqcup M_1} \overline{X}\right) &= -\frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} \beta_k \; \mathrm{res} \log \Delta_{k,Y_0} + \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} \beta_k \; \mathrm{res} \log \Delta_{k,Y_1} \\ &= -\log T^{\mathrm{res},\beta}_{Y_0} + \log T^{\mathrm{res},\beta}_{Y_1}. \qquad (6.3.2) \end{aligned}$$

THEOREM 6.3.1. (6.3.1) is a LogTQFT.

PROOF. We only have to check log-additivity, which follows in a straightforward way from the additivity of $\mathrm{res}\left(\log^{\beta}_{M_0 \sqcup M_1} \overline{X}\right)$ and the fact that res is the unique trace for $\Psi^{\mathbb{Z}}$. The additivity of $\mathrm{res}\left(\log^{\beta}_{M_0 \sqcup M_1} \overline{X}\right)$ is also straightforward thanks to the strategic choice of the sign.

□

REMARK 6.3.2. $\mathrm{res}\left(\log^{\beta}_{M_0 \sqcup M_1} \overline{X}\right)$ is non trivial only if n is odd (and hence dim is even). Also, the log-determinants (6.3.2) equal the homotopy invariants $\chi(M_1) - \chi(M_0)$ if $\beta = \underline{1}$ or $\chi'(M_1) - \chi'(M_0)$ if $\beta = \underline{\omega}$.

If we restrict to the category of h-cobordisms h-$\mathbf{Cob}_n$, then we can consider the character arising from the zeta trace. By h-$\mathbf{Cob}_n$ we mean a category whose objects are $\mathrm{obj}(\mathbf{Cob}_n)$ and whose morphisms $\overline{W} \in \mathrm{mor}_{h\text{-}\mathbf{Cob}_n}(M_0, M_1)$, called an *$h$-cobordism*, are cobordisms $\overline{W} \in \mathrm{mor}_{\mathbf{Cob}_n}(M_0, M_1)$ for which the inclusions $\iota_i : M_i \to W$ are homotopy equivalences (or, equivalently, such that M_i are deformation retracts of W).

REMARK 6.3.3. If we want to obtain smooth invariants, we will need acyclicity. Thus, the objects should be considered a pairs (M, ρ) where $\rho : \pi_1(M) \to O(N)$ is an acyclic representation (generating the flat associate bundle E_ρ). In this way,

$\pi_1(W) = \pi_1(M_0) = \pi_1(M_1) =: \pi_1$ and for every composable h-cobordism we will have the same orthogonal representation $\rho : \pi_1 \to O(N)$ (and thus the same coefficient bundle E_ρ) to consider.

Therefore, let us consider a log-functor defined as (6.3.1), but now restricted to h-$\mathbf{Cob}_n$, i.e. $\log : \mathcal{N}h\text{-}\mathbf{Cob}_n \to F_{\mathbb{Z},\Pi}(h\text{-}\mathbf{Cob}_n^*)$. Then, with respect to the zeta trace, we obtain as character:

$$\begin{aligned}\mathrm{TR}_\zeta\left(\log^\beta_{M_0\sqcup M_1}\overline{X}\right) &= -\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \mathrm{TR}_\zeta\log\Delta_{k,Y_0}\\ &\quad+\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \mathrm{TR}_\zeta\log\Delta_{k,Y_1}\\ &= -\log T^\beta_{Y_0}(\rho)+\log T^\beta_{Y_1}(\rho),\end{aligned}$$

which is, for $\beta = \underline{\omega}$, the difference of the analytic torsions of the boundary components. The latter coincides, for n even, with $\log\det\tau^{\mathrm{Wh}}(X)$, where $\tau^{\mathrm{Wh}}(X)$ is the *Whitehead torsion* of X (see §3.4, [**72**] for details).

REMARK 6.3.4. When restricted to h-$\mathbf{Cob}_n$, the character $\operatorname{res}\left(\log^\beta_{M_0\sqcup M_1}\overline{X}\right)$ is always trivial for $\beta = \underline{1}$ or $\beta = \underline{\omega}$. In fact, it would depend on the difference $\chi(M_1) - \chi(M_0)$, which is always vanishing when M_0 and M_1 are homotopically equivalent.

6.4. Residue Analytic Torsion for families

We recall from §4.4 that if $M \hookrightarrow \mathcal{M} \to B$ is a smooth fibre bundle with closed fibre $M \cong M_b$, $b \in B$, and $\mathcal{E} \to \mathcal{M}$ is a family of vector bundles with *flat* connection $\nabla^{\mathcal{E}}$, we have a natural family of de Rham operators $\mathscr{D} = d^M + \delta^M$ acting on $\Omega_{\mathrm{vert}}(\mathcal{M},\mathcal{E}) \cong C^\infty(B,\mathcal{W})$ (recall that $\mathcal{W} := \pi_*(\Lambda_\pi(\mathcal{M})\otimes\mathcal{E}) \to B$) and a family of Hodge Laplacians $\Delta^M := (d^M+\delta^M)^2 \in \Psi^2_{\mathrm{vert}}(\mathcal{M},\Lambda_\pi(\mathcal{M})\otimes\mathcal{E})$.

Together with the families of exterior derivatives and coderivatives, we also have the natural exterior derivative over the total space $\mathcal{M}$, $d^{\mathcal{M}} : \Omega(\mathcal{M},\mathcal{E}) \to \Omega(\mathcal{M},\mathcal{E})$.

PROPOSITION 6.4.1 (Proposition 3.4, [**9**]). $d^{\mathcal{M}}$ is a flat superconnection of total degree 1 on $\Lambda_\pi(\mathcal{M})\otimes\mathcal{E}$ such that

$$d^{\mathcal{M}} = d^M + \nabla^{\mathcal{W}} + i_T, \tag{6.4.1}$$

where $i_T \in \Omega^2(B,\mathrm{Hom}(\mathcal{W}^\bullet,\mathcal{W}^{\bullet-1}))$ is a 2-form (which depends on the *curvature* T of the fibre bundle).

The family of Hodge operators defining δ^M also defines an adjoint superconnection

$$\delta^{\mathcal{M}} = \delta^M + (\nabla^{\mathcal{W}})^* - T \wedge \cdot \tag{6.4.2}$$

(Proposition 3.7, [**9**]). Together with $d^{\mathcal{M}}$, we obtain the superconnection $d^{\mathcal{M}}+\delta^{\mathcal{M}} \in \mathcal{A}(B, \Psi^1(\mathcal{M}, \Lambda_\pi(\mathcal{M}) \otimes \mathcal{E}))$ (Proposition 3.9, [**9**]) adapted to the family of de Rham operators $d^M + \delta^M$ (Definition 4, [**62**]). In the same way, the Laplacian over $\mathcal{M}$, $\Delta^{\mathcal{M}} := d^{\mathcal{M}} + \delta^{\mathcal{M}} : \Omega(\mathcal{M}, \mathcal{E}) \to \Omega(\mathcal{M}, \mathcal{E})$, is adapted to the family of Laplacians $\Delta^M : \Omega_{\mathrm{vert}}(\mathcal{M}, \mathcal{E}) \to \Omega_{\mathrm{vert}}(\mathcal{M}, \mathcal{E})$.

Let $H^*(M,E)_b = \bigoplus_{i=0}^{\dim M} H^i(M_b, E_b)$ be the cohomology of $(\Omega(M_b, E_b), d_b)$. It is the fibre of a $\mathbb{Z}$-graded vector bundle $H^*(M,E) \to B$, the *cohomology bundle* of $\mathcal{W} \to B$ (Definition 3.13, [**9**]). Since $\mathcal{E} \to \mathcal{M}$ is flat, the Chern character of $H^*(M,E)$, $\mathrm{ch}(H^*(M,E)) \in H^*(B,\mathbb{R})$, actually corresponds to $\mathrm{rk}(\mathcal{E})\chi(M) \in \mathbb{Z}$. By Hodge Theory,

$$H^*(M,E)_b \cong \ker(d_b + \delta_b) \cong \ker(\Delta_b),$$

which assures the existence of $\mathbb{Z}$-graded vector bundles $\ker(d^M + \delta^M) \to B$ and $\ker(\Delta^M) \to B$, with $H^*(M,E) \cong \ker(d^M + \delta^M) \cong \ker(\Delta^M)$. Let Π_{H^*} denote the projection of $\Omega_{\mathrm{vert}}(\mathcal{M}, \mathcal{E})$ onto $H^*(M,E)$.

REMARK 6.4.2. For $\mathcal{Q} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ there is a natural notion of classical symbol (with differential form coefficients) and, when $\mathcal{Q}$ is invertible and admissible with spectral cut θ (Definition 4.3.5), one can define complex powers and logarithms as for the single operator case:

i) $\mathcal{Q}_\theta^{-s} = \frac{i}{2\pi} \int_{\mathscr{C}} \lambda_\theta^{-s} (\mathcal{Q} - \lambda I)^{-1} d\lambda \in \mathcal{A}(B, \Psi(\mathcal{M}, \mathcal{E}))$ (Lemma 1, [**62**]);

ii) $\log_\theta \mathcal{Q} = \frac{d}{ds}_{|s=0} \mathcal{Q}_\theta^s$ and $(\log_\theta \mathcal{Q})_{[0]} = \log_\theta \mathcal{Q}_{[0]} \in \Psi^0_{\mathrm{vert}}(\mathcal{M}, \mathcal{E})$ (Lemma 2, [**62**]).

If $\mathcal{Q}$ is not invertible, then $\mathcal{Q} + \Pi_{\mathcal{Q}_{[0]}}$ is so, where $\Pi_{\mathcal{Q}_{[0]}}$ is the orthogonal projection onto $\ker \mathcal{Q}_{[0]}$, which is a well-defined vector bundle over B if we assume $\dim\ker(\mathcal{Q}_{[0]})_b$ constant. In this case, $\log \mathcal{Q} := \log_\theta(\mathcal{Q} + \Pi_{\mathcal{Q}_{[0]}})$.

REMARK 6.4.3. For a family of ψdo-valued forms $\mathcal{Q} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ it is possible also to define a Wodzicki residue trace and a zeta-trace in a natural way (§3, [**62**]). In particular, there exists a well-defined residue trace density $\mathrm{res}_x(\mathcal{Q}) \in C^\infty(\mathcal{M}, \pi^*\Lambda(B))$ (which is defined in analogy to the single operator case), and via integration along the fibre a residue trace:

$$\mathrm{res}(\mathcal{Q}) := \int_{\mathcal{M}/B} \mathrm{res}_x(\mathcal{Q})dx \in \Omega^*(B).$$

If also $\mathcal{Q} = \sum_{k=0}^{\dim B} \mathcal{Q}_{[k]} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ satisfies $\mathrm{ord}\mathcal{Q}_{[0]} = q_0 > 0$ and

$$q_k \leq q_0 \qquad \forall k = 0, \ldots, \dim B, \qquad q_k := \mathrm{ord}\mathcal{Q}_{[k]}, \tag{6.4.3}$$

then a res-logarithm can be defined (§3, **[62]**):

$$\mathrm{res}\log\mathcal{Q} := \int_{\mathcal{M}/B} \mathrm{res}_x(\log\mathcal{Q})dx \in \Omega^*(B).$$

THEOREM 6.4.4 (From Theorem 3, **[62]**). Let $\mathcal{Q} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ be admissible and satisfy (6.4.3). Assume also that $\ker\mathcal{Q}_{[0]} \to B$ is a well-defined vector bundle and consider $\mathcal{R} \in \mathcal{A}(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ such that $\mathcal{R}_{[k]}$ is a differential operator for each k. Then:

$$-\frac{1}{q_0}\mathrm{res}\left(\mathcal{R}\log\mathcal{Q}\right) = \zeta(\mathcal{R}, \mathcal{Q}, 0)|^{\mathrm{mer}} + \mathrm{tr}\left(\mathcal{R}\Pi_{\ker\mathcal{Q}_{[0]}}\right) \in \Omega^*(B).$$

REMARK 6.4.5 (§4, **[62]**). A ζ-regularization is clearly well-defined also in this family setting, hence giving rise to a meromorphic map $\zeta(\mathcal{R}, \mathcal{Q}, 0)|^{\mathrm{mer}}$. Moreover, as for the single operator case, ζ, res and TR_ζ for families are related by the same formulas.

This applies to superconnections, thus yielding:

THEOREM 6.4.6 (From Theorem 4, **[62]**). Let $\mathcal{Q}$ be a superconnection adapted to a smooth family of formally self-adjoint elliptic pseudodifferential operators $\mathcal{P} = \mathcal{Q}_{[0]} \in \mathcal{A}^0(B, \Psi^m(\mathcal{M}, \mathcal{E}))$ satisfying (6.4.3). Assume also that $\ker\mathcal{Q}_{[0]} \to B$ is a well-defined vector bundle. Then:

$$-\frac{1}{2q_0}\mathrm{res}\left(\mathcal{Q}^{2k}\log\mathcal{Q}^2\right) = \zeta(\mathcal{Q}^{2k}, \mathcal{Q}^2, 0)|^{\mathrm{mer}} + \mathrm{tr}\left(\mathcal{Q}^{2k}\Pi_{\ker\mathcal{Q}_{[0]}}\right) \in \Omega^*(B)$$

is closed and $\zeta(\mathcal{Q}^{2k}, \mathcal{Q}^2, 0)|^{\mathrm{mer}}$ is exact. Therefore:

$$-\frac{1}{2q_0}\mathrm{res}\left(\mathcal{Q}^{2k}\log\mathcal{Q}^2\right) = \mathrm{tr}\left(\mathcal{Q}^{2k}\Pi_{\ker\mathcal{Q}_{[0]}}\right) \in H^*(B, \mathbb{R})$$

COROLLARY 6.4.7. $-\frac{1}{2q_0}\mathrm{res}\log\mathcal{Q}^2 = \mathrm{tr}\left(\Pi_{\ker\mathcal{Q}_{[0]}}\right) \in \mathbb{Z}$

PROOF. As $\ker\mathcal{Q}_{[0]} \to B$ is assumed to be a well-defined vector bundle, the function $b \mapsto \mathrm{tr}\left(\Pi_{\ker\mathcal{Q}_{[0],b}}\right)$ is locally constant.

∎

DEFINITION 6.4.8. Given an $(n+1)$-tuple $\beta = (\beta_0, \dots, \beta_n) \in \mathbb{R}^{n+1}$, we define the *family torsion logarithm* to be the operator:

$$\mathbb{T}_{\mathcal{M}} := \frac{1}{2} \sum_{k=0}^{\dim M} (-1)^{k+1} \beta_k \log \Delta_k^{\mathcal{M}} \in \mathcal{A}(B, \Psi^{\mathbb{Z}}(\mathcal{M}, \mathcal{E}))$$

The *family analytic residue torsion* $\mathscr{T}_{\mathcal{M}}^{\mathrm{res},\beta}$ is the class in $H^*(B,\mathbb{R})$ of the character of $\mathbb{T}_{\mathcal{M}}$ with respect to the residue trace, i.e.

$$\mathscr{T}_{\mathcal{M}}^{\mathrm{res},\beta} := \mathrm{res}\ \mathbb{T}_{\mathcal{M}} = \frac{1}{2} \sum_{k=0}^{\dim M} (-1)^{k+1} \beta_k \mathrm{res} \log \Delta_k^{\mathcal{M}} \in H^*(B,\mathbb{R}).$$

THEOREM 6.4.9. Let $\mathcal{M} \to B$ be a fibre bundle with closed oriented n-dimensional fibre M and $\mathcal{E} \to \mathcal{X}$ flat Hermitian vector bundle. Then:

i) if n is odd, $\mathscr{T}_{\mathcal{M}}^{\mathrm{res},\beta} = 0\ \forall \beta \in \mathbb{R}^{n+1}$;

ii) if n is even, $\mathscr{T}_{\mathcal{M}}^{\mathrm{res},\beta}$ is a smooth invariant if β equals:

$$\underline{1} = (1, \dots, 1) \quad \text{or} \quad \underline{\omega} = (0, 1, \dots, n). \tag{6.4.4}$$

The corresponding family residue analytic torsions are the Euler characteristic and derived Euler characteristic of the fiber:

$$\mathscr{T}_{\mathcal{M}}^{\mathrm{res},\underline{1}} = \chi(X,E) \qquad \text{and} \qquad \mathscr{T}_{\mathcal{M}}^{\mathrm{res},\underline{\omega}} = \chi'(X,E).$$

Finally, for a smooth path of vertical metrics $u \mapsto g^{\mathcal{M}/B}(u)$ we have

$$\frac{d}{du} \mathscr{T}_{\mathcal{M}}^{\mathrm{res},\underline{\omega}}(u) = \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} \underbrace{\mathrm{res} \Lambda_k}_{=0}, \qquad \Lambda_k := (\alpha_{k,b})_{b \in B}.$$

PROOF. (i) By (6.4.1) and (6.4.2), $d^{\mathcal{M}} + \delta^{\mathcal{M}}$ satisfies (6.4.3) and is a smooth form with differential operator coefficients. Then $\log \Delta^{\mathcal{M}} - \log \Delta^{M} \in \mathcal{A}(B, \Psi^{\mathbb{Z}}(\mathcal{M}, \mathcal{E}))$ (Lemma 2, [**62**]), and hence $\log \Delta^{\mathcal{M}}$, is a sum of forms whose coefficients are logarithms of differential operator, hence odd-class and thus the integration of the fibre of its residue density vanishes in odd (fibre) dimension, as in the proof of (i) Theorem 6.2.28.

(ii) If n is even, then the proof works as for the single operator case, fiberwise. In fact, the change in the metric generates the vertical multiplication operator Λ_k for which the family Wodzicki residue vanishes, as explained in §3 of [**62**].

□

REMARK 6.4.10. As for the single manifold case, we can define a LogTQFT (this time a higher one) from the family residue torsion:

$$\log : \mathcal{N}\mathbf{FCob}_n(B) \to HC_*\left(F(\mathbf{FCob}_n^*)\right)$$

with $\log_{\mathcal{M}_0 \sqcup \mathcal{M}_1} \overline{\mathcal{W}} := \mathscr{T}^{\mathrm{res},\beta}_{\mathcal{M}_0} - \mathscr{T}^{\mathrm{res},\beta}_{\mathcal{M}_1} \in H^0(B)$. Its properties are easy to check and it represents a rather simple higher LogTQFT, as its higher order terms are all zero.

6.5. Manifolds with boundary

6.5.1. Analytic Torsion of manifolds with boundary. When X has a non-empty boundary Y, Green's formula yields:

(6.5.1)
$$\langle \Delta\omega, \theta\rangle_X = \langle \omega, \Delta\theta\rangle_X + \int_Y \omega \wedge *d\theta - \int_Y \theta \wedge *d\omega + \int_Y \delta\omega \wedge *\theta - \int_Y \delta\theta \wedge *\omega,$$

for $\omega, \theta \in \Omega(X, E_\rho)$ ((2.8), [**18**]). Hence, $\Delta : \Omega(X, E_\rho) \to \Omega(X, E_\rho)$ becomes self-adjoint when relative or absolute boundary conditions are imposed.

Definition 6.5.1 (§2.1, [**18**]). *Relative* and *absolute boundary conditions*, respectively, for $\Delta = d\delta + \delta d$ are defined as:

Relative: $\begin{cases} \mathcal{R}\gamma\omega = 0 \\ \mathcal{R}\gamma(d+\delta)\omega = \mathcal{R}\gamma\delta\omega = 0 \end{cases}$ Absolute: $\begin{cases} \mathcal{A}\gamma\omega = 0 \\ \mathcal{A}\gamma(d+\delta)\omega = \mathcal{A}\gamma d\omega = 0 \end{cases}$

Its realization $\Delta_{\mathcal{R}}$ is the L^2-closure of an unbounded operator acting like Δ and with domain $\{\omega \in \Omega(X, E_\rho) | \mathcal{R}\gamma\omega = 0, \mathcal{R}\gamma\delta\omega = 0\}$. When absolute boundary conditions will be considered, then we will write $\Delta_{\mathcal{A}}$.

Remark 6.5.2 (§7, [**12**]). Relative, resp. absolute, boundary conditions are equivalent to:

Relative: $\begin{cases} \mathcal{R}\gamma\omega = 0 \\ \mathcal{A}\gamma\partial_t\omega = 0 \end{cases}$ Absolute: $\begin{cases} \mathcal{A}\gamma\omega = 0 \\ \mathcal{R}\gamma\partial_t\omega = 0, \end{cases}$

i.e. are *normal* (according to the terminology in §3.3, [**26**]). For second order operators, this stands for boundary conditions of the form

$$T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} : C^\infty(X, E) \to C^\infty(Y, E') \oplus C^\infty(Y, E'), \text{ with}$$
$$T_0 = s_0(y)\gamma + T_0' \quad \text{and} \quad T_1 = s_1(y)\gamma\partial_t + S_{1,0}\gamma + T_1',$$

with $s_0(y)$ and $s_1(y)$ surjective endomorphisms. For example, $s_0(y) = \mathcal{R}$, $s_1(y) = \mathcal{A}$, and $T_0' = S_{1,0} = T_1' = 0$ for relative boundary conditions.

With relative, resp. absolute, boundary conditions, (6.5.1) yields that Δ becomes self-adjoint. Therefore the realization $\Delta_{k,\mathcal{R}}$, resp. $\Delta_{k,\mathcal{A}}$ has a discrete set of non-negative eigenvalues accumulating at infinity and a corresponding orthonormal basis of eigenvalues for $L^2(X,E)$, which satisfy the boundary conditions (for instance, by Lemma 1.9.1, [**23**]).

Each $\Delta_{k,B}$ has $\mathbb{R}^-$ as a ray of minimal growth, i.e. $\theta=\pi$ is an Agmon angle for $\Delta_{k,B}$. Therefore, for $\Re(s)>0$ and $\mathscr{C}$ the Laurent Loop (6.2.6),

$$\Delta_{k,B}^{-s} := \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-s}\ (\Delta_{k,B}-\lambda)^{-1}d\lambda,$$

is a holomorphic family ([**77**], [**78**]), and is trace class for $\Re(s)>n/2$ ([**77**]). Therefore, if Tr is the classical trace, $\mathrm{Tr}(\Delta_{k,B}^{-s})$ is holomorphic for $\Re(s)>n/2$ and by linearity of Tr:

$$\frac{d}{ds}\mathrm{Tr}(\Delta_{k,B}^{-s}) = \mathrm{Tr}(\frac{d}{ds}\Delta_{k,B}^{-s}) \quad \text{ for } \Re(s)>n/2.$$

$\mathrm{Tr}(\Delta_{k,B}^{-s})$ and $\mathrm{Tr}(\frac{d}{ds}\Delta_{k,B}^{-s})$ can be extended meromorphically and are holomorphic at $s=0$ (by expansion (1.12), [**28**]). Therefore, if we define the zeta function to be the meromorphic extension $\zeta_{k,B}(s) := \mathrm{Tr}(\Delta_{k,B}^{-s})|^{\mathrm{mer}}$, we obtain

$$\frac{d}{ds}\zeta_{k,B}(s) = \mathrm{Tr}(\frac{d}{ds}\Delta_{k,B}^{-s})|^{\mathrm{mer}} = -\mathrm{Tr}(\log\Delta_{k,B}\cdot\Delta_{k,B}^{-s})|^{\mathrm{mer}},$$

where:

$$\log\Delta_{k,B} := \lim_{s\searrow 0}\frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\lambda^{-s}\ (\Delta_{k,B}-\lambda)^{-1}d\lambda$$

has been defined by Grubb and Gaarde, (2.5) in [**22**]. Therefore by [**28**] we can conclude:

Lemma 6.5.3.

$$\frac{d}{ds}\zeta_{\Delta_{k,B}}(0) = -\mathrm{TR}_\zeta(\log\Delta_{k,B}), \tag{6.5.2}$$

for TR_ζ the generalization of the ζ-trace to Boutet de Monvel calculus, discussed in [**28**].

Remark 6.5.4 (§2.2, [**18**]; §7, [**65**]). The spectral zeta function of $\Delta_{k,\mathcal{B}}$, with $\mathcal{B}$ either $\mathcal{R}$ or $\mathcal{A}$, is also equivalently defined as:

$$\zeta_{k,\mathcal{B}}(s) := \zeta(\Delta_{k,\mathcal{B}},s) = \sum_{\lambda_i}\lambda_i^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty\ t^{s-1}\ \mathrm{Tr}\left(e^{t\Delta_{\mathcal{B}}^k}(I-\Pi_k)\right)dt$$

for $\lambda\notin\mathrm{spec}(\Delta_{\mathcal{B}})$, $e^{t\Delta_{k,\mathcal{B}}}$ the heat operator associated to $\Delta_{k,\mathcal{B}}$, and Π_k the orthogonal projection onto the generalized $\ker(\Delta_{k,\mathcal{B}})$. The generalization to $\zeta(A,\Delta_{k,\mathcal{B}},s)$ is straightforward.

LEMMA 6.5.5.

$$\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) = (-1)^{n-1}\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{A}}(s). \tag{6.5.3}$$

PROOF. Since $\Delta_{k,\mathcal{R}}$, resp. $\Delta_{k,\mathcal{A}}$ has a discrete set of non-negative eigenvalues accumulating at infinity, the proof follows the one in the closed case, i.e. the proof of Theorem 2.3, [**65**]. Let $\lambda \neq 0$ an eigenvalue for $\Delta_{k,\mathcal{R}}$ and denote by

$$\mathcal{E}_{k,\mathcal{R}}(\lambda) = \{\omega \in \Omega^k(X, E_\rho)|\ \Delta\omega = \lambda\omega, \mathcal{R}\gamma\omega = \mathcal{R}\gamma\delta\omega = 0\}$$

the associated eigenspace. Then:

$$\Lambda'_k(\lambda) = \frac{1}{\lambda}d\delta \qquad \text{and} \qquad \Lambda''_k(\lambda) = \frac{1}{\lambda}\delta d$$

are orthogonal projections of $\mathcal{E}_{k,\mathcal{R}}(\lambda)$ onto $\mathcal{F}_{k,\mathcal{R}}(\lambda) = \{\omega \in \mathcal{E}_{k,\mathcal{R}}(\lambda)|\ d\omega = 0\}$ and $\mathcal{G}_{k,\mathcal{R}}(\lambda) = \{\omega \in \mathcal{E}_{k,\mathcal{R}}(\lambda)|\ \delta\omega = 0\}$, respectively. Also, by construction, $\Lambda'_k(\lambda) + \Lambda''_k(\lambda) = I$. Since the map $\frac{1}{\sqrt{\lambda}}d$ is an isomorphism with inverse $\frac{1}{\sqrt{\lambda}}\delta$, we conclude $\mathcal{G}_{k,\mathcal{R}}(\lambda) \cong \mathcal{F}_{k+1,\mathcal{R}}(\lambda)$ and thence:

$$g_{k,\mathcal{R}}(\lambda) = |\mathcal{G}_{k,\mathcal{R}}(\lambda)| = |\mathcal{F}_{k+1,\mathcal{R}}(\lambda)| = f_{k+1,\mathcal{R}}(\lambda).$$

Therefore:

$$\begin{aligned}\zeta_{k,\mathcal{R}}(s) &= \sum_{\lambda\neq 0}\lambda^{-s}|\mathcal{E}_{k,\mathcal{R}}(\lambda)| = \sum_{\lambda\neq 0}\lambda^{-s}(f_{k,\mathcal{R}}(\lambda) + f_{k+1,\mathcal{R}}(\lambda))\\ &= \sum_{\lambda\neq 0}\lambda^{-s}(g_{k,\mathcal{R}}(\lambda) + g_{k-1,\mathcal{R}}(\lambda)) \qquad \text{and}\end{aligned}$$

$$\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) = \sum_{k=1}^{n}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}f_{k,\mathcal{R}}(\lambda) = -\sum_{k=0}^{n-1}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}g_{k,\mathcal{R}}(\lambda).$$

By Proposition 0.3.3, $*\mathcal{R} = \mathcal{A}*$, which yields $\mathcal{F}_{k,\mathcal{R}}(\lambda) \cong \mathcal{G}_{n-k,\mathcal{A}}(\lambda)$ and therefore $f_{k,\mathcal{R}}(\lambda) = g_{n-k,\mathcal{A}}(\lambda)$ and

$$\zeta_{k,\mathcal{R}}(s) = \zeta_{n-k,\mathcal{A}}(s). \tag{6.5.4}$$

In conclusion:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) &= -\sum_{k=0}^{n-1}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}g_{k,\mathcal{R}}(\lambda) = \sum_{k=1}^{n}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}f_{k,\mathcal{R}}(\lambda)\\ &= \sum_{k=1}^{n}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}g_{n-k,\mathcal{A}}(\lambda) = \sum_{k=0}^{n-1}(-1)^{n-k}\sum_{\lambda\neq 0}\lambda^{-s}g_{k,\mathcal{A}}(\lambda)\\ &= (-1)^n\sum_{k=0}^{n-1}(-1)^k\sum_{\lambda\neq 0}\lambda^{-s}g_{k,\mathcal{A}}(\lambda) = (-1)^{n-1}\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{A}}(s).\end{aligned}$$

□

REMARK 6.5.6. It came to our attention that (6.5.3) had been proven by W. Lück, [**47**], Proposition 2.10 (unsurprisingly called *Poincaré duality for analytic torsion*). We stress the fact that the approach is very similar and is based on the proof of Theorem 2.3, [**65**]. This latter result can be obtained as a corollary for n even and $Y = \emptyset$, since in this case $\zeta_{k,\mathcal{R}}(s) = \zeta_{k,\mathcal{A}}(s) = \zeta_k(s)$, tautologically.

COROLLARY 6.5.7.

$$\sum_{k=0}^{n}(-1)^k \zeta_{k,\mathcal{R}}(s) = \sum_{k=0}^{n}(-1)^k \zeta_{k,\mathcal{A}}(s) = 0.$$

PROOF. Since $\zeta_{k,\mathcal{R}}(s) = \zeta_{n-k,\mathcal{A}}(s)$, from (6.5.3) we have:

$$\begin{aligned}
0 &= \sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) + (-1)^n \sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{A}}(s) \\
&= \sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) + (-1)^n \sum_{k=0}^{n}(-1)^k k\zeta_{n-k,\mathcal{R}}(s) \\
&= \sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) + (-1)^n \sum_{k=0}^{n}(-1)^{n-k} (n-k)\zeta_{k,\mathcal{R}}(s) \\
&= \sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s) + \sum_{k=0}^{n}(-1)^{k} (n-k)\zeta_{k,\mathcal{R}}(s) \\
&= n\sum_{k=0}^{n}(-1)^k \zeta_{k,\mathcal{R}}(s) = (-1)^n n \sum_{k=0}^{n}(-1)^k \zeta_{k,\mathcal{A}}(s)
\end{aligned}$$

□

REMARK 6.5.8. Unlike for even dimensional closed manifolds, $\sum_{k=0}^{n}(-1)^k k\zeta_{k,\mathcal{R}}(s)$ may not vanish in general, as we can see from the following examples.

Let $X = [0, R]$ (i.e. $n = 1$ and $Y = \{0\} \sqcup \{R\}$); the eigenvalue problem for $\Delta_0 = -\partial_x^2$ with relative boundary conditions is just the harmonic oscillator with Dirichlet boundary conditions. As it is well-known, its eigenvalues are $\lambda = \frac{n^2\pi^2}{R^2}$, $n \in \mathbb{N}$, and therefore:

$$\zeta_{0,\mathcal{R}}(s) = 2\frac{R^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty} n^{-2s} = 2\frac{R^{2s}}{\pi^{2s}} \zeta_{\mathbf{R}}(2s),$$

where $\zeta_{\mathbf{R}}(s)$ is the *Riemann zeta function*. Consequently,

$$\sum_{k=0}^{1}(-1)^k k\zeta_{k,\mathcal{A}}(s) = -\zeta_{1,\mathcal{A}}(s) \overset{(6.5.4)}{=} -\zeta_{0,\mathcal{R}}(s) = -2\frac{R^{2s}}{\pi^{2s}}\zeta_{\mathbf{R}}(2s)$$

does not vanish identically and $\sum_{k=0}^{1}(-1)^k k\zeta_{k,\mathcal{A}}(0) = -2\zeta_{\mathbf{R}}(0) = 1$.

Analogously, let now X be the cylinder $[0, R] \times S^1$, with $x \in [0, R]$ the normal coordinate; hence, $\Delta = -\partial_x^2 + \Delta^{S^1}$ and $\zeta_{1,\mathcal{R}}(s) = \zeta_{0,\mathcal{R}}(s) + \zeta_{2,\mathcal{R}}(s)$ by Corollary

6.5.7. Therefore,

$$\sum_{k=0}^{2}(-1)^k k\zeta_{k,\mathcal{R}}(s) = -\zeta_{1,\mathcal{R}}(s) + 2\zeta_{2,\mathcal{R}}(s) = \zeta_{2,\mathcal{R}}(s) - \zeta_{0,\mathcal{R}}(s) = \zeta_{0,\mathcal{A}}(s) - \zeta_{0,\mathcal{R}}(s).$$

Since Δ_0 with relative/absolute boundary conditions corresponds to the Laplacian on functions with Dirichlet/Neumann conditions, we have $\zeta_{0,\mathcal{A}}(s) - \zeta_{0,\mathcal{R}}(s) = \zeta_0^{S^1}(s)$ (§3.2, [**37**]). In particular, by an easy calculation one obtains that $\zeta_0^{S^1}(s) = 2\zeta_{\mathbf{R}}(2s)$.

As in the closed manifold case, once a notion of zeta function holomorphic at zero is established, one can define the analytic torsion.

Definition 6.5.9 (7.2, [**65**]). Let X be a manifold with non-empty boundary and $\rho : \pi_1(X) \to O(N)$ an orthogonal representation. Then the *analytic torsion* with relative boundary conditions $T_{X,\mathcal{R}}(\rho)$ of X is defined as:

$$\log T_{X,\mathcal{R}}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k k \frac{d}{ds}\zeta_{k,\mathcal{R}}(0).$$

The analytic torsion with absolute boundary conditions $T_{X,\mathcal{A}}(\rho)$ is analogously defined.

Remark 6.5.10 ([**18**]). By (6.5.3), $\log T_{X,\mathcal{A}}(\rho) = (-1)^{n-1}\log T_{X,\mathcal{R}}(\rho)$.

Vishik [**83**] generalized Cheeger-Müller theorem to:

Theorem 6.5.11 (1.4, [**83**]). $X = X_1 \cup_{Y_1} X_2$ and $Y = \partial X$

$$T_X(\rho) = 2^{\frac{\chi(Y)}{2}}\tau_X(\rho) \text{ and } \log T_{X_1\cup_{Y_1}X_2}(\rho) = 2^{\frac{\chi(Y)}{2}+\chi(Y_1)}\tau_X(\rho).$$

6.5.2. Analytic Residue Torsion of manifolds with boundary. From (6.5.2) we can rewrite the analytic torsion as:

$$\log T_{X,B}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}k\mathrm{TR}_\zeta(\log\Delta_{k,B}), \qquad B = \mathcal{R} \text{ or } \mathcal{A}.$$

Hence once again our analysis shifts to the more fundamental invariant:

$$\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}k\log\Delta_{k,B},$$

which now belongs to the Bouted de Monvel calculus (from [**22**]). There, the residue trace has been exteded by work of Fedosov, Golse, Leichtnam, and Schrohe (we only refer to [**21**] for the definition and a detailed exposition) and is the unique trace of this algebra. Hence, we have a well-defined $\mathrm{res}\log\Delta_{k,B}$, which we can use to define a (generalized) residue analytic torsion of X with either relative or absolute boundary conditions:

$$\log T^{\mathrm{res},\beta}_{X,B}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\beta_k \ \mathrm{res}\log\Delta_{k,B}, \tag{6.5.5}$$

where B stands for either $\mathcal{R}$ or $\mathcal{A}$.

THEOREM 6.5.12. Let X be an oriented manifold with boundary Y. Then $\log T^{\mathrm{res},\beta}_{X,B}(\rho)$ is a smooth invariant if β equals

$$\underline{1} = (1,\dots,1) \quad \text{or} \quad \underline{\omega} = (0,1,\dots,n).$$

The corresponding residue analytic torsions are:

$$\log T^{\mathrm{res},\underline{1}}_{X,B}(\rho) = \chi_B(X,E_\rho) \quad \text{and} \quad \log T^{\mathrm{res},\underline{\omega}}_{X,B}(\rho) = \chi'_B(X,E_\rho) + \sum_{k=0}^{n}(-1)^k k\zeta_{k,B}(0).$$

Finally, for a smooth path of metrics $[0,1] \ni u \mapsto g^X(u)$ for which the normal direction to the boundary is the same, we have:

$$\frac{d}{du}\log T^{\mathrm{res},\underline{\omega}}_{X,B}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} \ \mathrm{res}\ \alpha_k.$$

PROOF. For the proof, we follow the idea of Remark 6.2.31, almost identically. In fact, by **[27]**, we have that:

$$-\frac{1}{2}\mathrm{res}\log\Delta_{k,B} = \zeta_{k,B}(0) + \dim\ker\Delta_{k,B}, \tag{6.5.6}$$

as relative/absolute boundary conditions are normal. The claim will follow as for the closed case, since appropriate trace asymptotic expansions were established by Grubb and Vishik.

By (6.5.6), we can re-write (6.5.5) as:

$$\log T^{\mathrm{res},\beta}_{X,B}(\rho) = \sum_{k=0}^{n}(-1)^k\beta_k \ \zeta_{k,B}(0) + \sum_{k=0}^{n}(-1)^k\beta_k \ \dim\ker\Delta_{k,B}.$$

Let $[0,1] \ni u \mapsto g^X(u)$ be a smooth path of metrics for which the normal direction to the boundary Y is the same and consider $\frac{d}{du}\log T^{\mathrm{res},\beta}_{X,B}(\rho)$. Since $\ker\Delta_{k,B}$ is isomorphic to relative/absolute de Rham cohomology, it is independent of the metric (see for instance the proof of Proposition 6.4, **[65]**, or (2.5) in **[83]**) and the derivative reduces to:

$$\frac{d}{du}\log T^{\mathrm{res},\beta}_{X,B}(\rho) = \frac{d}{du}\sum_{k=0}^{n}(-1)^k\beta_k \ \zeta_{k,B}(0).$$

Therefore, without loss of generality in this context, we can consider $\Delta_{k,B}$ to be invertible. Again, $\zeta_{k,B}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta_{k,B}(u)})dt$ and we can study the derivative at $s=0$ of:

$$f(u,s) := \sum_{k=0}^{n}(-1)^k\beta_k\ \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta_{k,B}(u)})dt.$$

By Theorem 6.1, [**65**], $\frac{\partial}{\partial u}\mathrm{Tr}(e^{-t\Delta_{k,B}}) = -t\mathrm{Tr}((\delta\alpha d - d\alpha\delta + \alpha d\delta - \alpha\delta d)e^{-t\Delta_{k,B}})$ and, by the proof of Proposition 2.15, [**83**], we can differentiate under the integral sign, thus obtaining:

$$\begin{aligned}\frac{\partial}{\partial u}f(u,s) &= \sum_{k=0}^{n}(-1)^k\beta_k\ \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\partial}{\partial u}\mathrm{Tr}(e^{-t\Delta_{k,B}})dt\\ &= \sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \frac{1}{\Gamma(s)}\int_0^\infty t^{s}\mathrm{Tr}((\delta\alpha d - d\alpha\delta + \alpha d\delta - \alpha\delta d)e^{-t\Delta_{k,B}})dt.\end{aligned}$$

Moreover, from Theorem 7.3 of [**65**], $\mathrm{Tr}(d\alpha\delta e^{-t\Delta_{k,B}}) = \mathrm{Tr}(\alpha\delta d e^{-t\Delta_{k-1,B}})$ and $\mathrm{Tr}(\delta\alpha d e^{-t\Delta_{k,B}}) = \mathrm{Tr}(\alpha d\delta e^{-t\Delta_{k+1,B}})$. Thus, if we set $\theta_k := \mathrm{Tr}(\alpha\delta d e^{-t\Delta_{k,B}})$ and $\varphi_k := \mathrm{Tr}(\alpha d\delta e^{-t\Delta_{k,B}})$, we obtain:

$$\frac{\partial}{\partial u}f(u,s) = \sum_{k=0}^{n}(-1)^{k+1}\beta_k\ \frac{1}{\Gamma(s)}\int_0^\infty t^{s}(\varphi_{k+1} - \theta_k + \varphi_k - \theta_{k-1})dt,$$

exactly as in Remark 6.2.31. Therefore, we have to face the same calculation for the closed case, which we know yields:

$$\begin{aligned}\frac{\partial}{\partial u}f(u,s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^s\sum_{k=1}^{n}(-1)^k(\beta_{k+1} - 2\beta_k + \beta_{k-1})\varphi_k dt\\ &\quad + \frac{1}{\Gamma(s)}\sum_{k=0}^{n}(-1)^k(\beta_{k+1} - \beta_k)\int_0^\infty t^s\frac{d}{dt}\mathrm{Tr}\left(\alpha e^{-t\Delta_{k,B}}\right)dt\\ &= \sum_{k=1}^{n}(-1)^k(\beta_{k+1} - 2\beta_k + \beta_{k-1})\frac{1}{\Gamma(s)}\int_0^\infty t^s\mathrm{Tr}(\alpha d\delta e^{-t\Delta_{k,B}})dt\\ &\quad + s\sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\alpha e^{-t\Delta_{k,B}}\right)dt\\ &= \sum_{k=1}^{n}(-1)^k(\beta_{k+1} - 2\beta_k + \beta_{k-1})s\zeta(\alpha d\delta\Delta_{k,B}^{-1}, \Delta_{k,B}, s)\\ &\quad + \sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)s\zeta(\alpha, \Delta_{k,B}, s).\end{aligned}$$

By (1.14), [**28**]:

$$\begin{aligned}\frac{\partial}{\partial u}f(u,0) &= \frac{1}{2}\sum_{k=1}^{n}(-1)^k(\beta_{k+1} - 2\beta_k + \beta_{k-1})\mathrm{res}(\alpha_k d\delta\Delta_{k,B}^{-1})\\ &\quad + \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}(\beta_{k+1} - \beta_k)\mathrm{res}(\alpha_k).\end{aligned}$$

Since α_k is the usual multiplication operator, $\mathrm{res}(\alpha_k) = 0$, while the first term on the right hand side vanishes if $\beta_{k+1} - 2\beta_k + \beta_{k-1}$, as in the closed case. We remark that for $\beta = \underline{\omega}$ we have, once again:

$$\log T_{X,B}^{\mathrm{res},\underline{\omega}}(\rho) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}\underbrace{\mathrm{res}(\alpha_k)}_{=0}$$

Therefore, if $\beta = \underline{1}$,

$$\log T_{X,B}^{\mathrm{res},\underline{1}}(\rho) = \sum_{k=0}^{n}(-1)^k\zeta_{k,B}(0) + \sum_{k=0}^{n}(-1)^k \dim\ker(\Delta_{k,B}) = 0 + \chi_B(X, E_\rho)$$

by Corollary 6.5.7, where $\chi_B(X, E_\rho)$ is the relative/absolute Euler characteristic, while if $\beta = \underline{\omega}$,

$$\begin{aligned}\log T_{X,B}^{\mathrm{res},\underline{\omega}}(\rho) &= \sum_{k=0}^{n}(-1)^k k\ \zeta_{k,B}(0) + \sum_{k=0}^{n}(-1)^k k \dim\ker(\Delta_{k,B})\\ &= \sum_{k=0}^{n}(-1)^k k\ \zeta_{k,B}(0) + \chi_B'(X, E_\rho),\end{aligned}$$

where $\chi_B'(X, E_\rho)$ is the relative/absolute derived Euler characteristic.

$\square$

DEFINITION 6.5.13. The *absolute* and *relative derived Euler characteristics* are the integers defined as:

$$\chi'(X) := \sum_{k=0}^{n}(-1)^k k \dim H_{\mathcal{A}}^k(X) \quad \text{and} \quad \chi'(X,Y) := \sum_{k=0}^{n}(-1)^k k \dim H_{\mathcal{R}}^k(X).$$

By Poincaré Duality, we can obtain some straightforward identities, as follows.

THEOREM 6.5.14. $\chi'(X) + (-1)^n\chi'(X,Y) = n\chi(X)$.

PROOF. Let $b_{\mathcal{A}}^k = \dim H_{\mathcal{A}}^k(X)$ and $b_{\mathcal{R}}^k = \dim H_{\mathcal{R}}^k(X)$; then $b_{\mathcal{A}}^k = b_{\mathcal{R}}^{n-k}$ by Poincaré Duality, and:

$$\begin{aligned}\chi'(X) &= \sum_{k=0}^{n}(-1)^k k b_{\mathcal{A}}^k = \sum_{k=0}^{n}(-1)^k k b_{\mathcal{R}}^{n-k} = \sum_{k=0}^{n}(-1)^{n-k}(n-k) b_{\mathcal{R}}^k\\ &= (-1)^n n\sum_{k=0}^{n}(-1)^k b_{\mathcal{R}}^k + (-1)^{n-1}\sum_{k=0}^{n}(-1)^k k b_{\mathcal{R}}^k\\ &= (-1)^{n-1}\chi'(X,Y) + (-1)^n n\chi(X,Y) = (-1)^{n-1}\chi'(X,Y) + n\chi(X),\end{aligned}$$

where the last equality holds because $\chi(X) = (-1)^n\chi(X,Y)$.

$\square$

COROLLARY 6.5.15. If n is odd, then:

$$\chi'(X) = \chi'(X,Y) + \chi'(Y) + \frac{1}{2}\chi(Y).$$

PROOF. Since n is odd, $n-1$ is even and by Corollary 6.2.15 $\chi'(Y) = \frac{n-1}{2}\chi(Y)$. From $\chi(X) = \frac{1}{2}\chi(Y)$, we obtain $\chi'(Y) = (n-1)\chi(X)$. Hence, by the previous theorem, $\frac{n}{n-1}\chi'(Y) = n\chi(X) = \chi'(X) - \chi'(X,Y)$.

□

REMARK 6.5.16. Interestingly enough, $\chi'(X) = \chi'(X,Y) + \chi'(Y) + \frac{1}{2}\chi(Y)$, does not hold when n is even. To see this, let us consider $X = D^n$ the n-dimensional disc. Then $H_*(D^n, S^{n-1}) = H_*(S^n)$ (at least for $* > 0$, which is good enough for χ') and $H_*(D^n) = H_*(\{\mathrm{pt}\})$ by homotopy equivalence. Therefore, $\chi'(D^n) = \chi'(\{\mathrm{pt}\}) = 0$, $\chi'(D^n, S^{n-1}) = \chi'(S^n) = (-1)^n n$, and $\chi(S^{n-1}) = 1 + (-1)^{n-1}$, which do not fit in the equation unless n is odd.

Finally, we have log-additivity of the the residue analytic torsion:

THEOREM 6.5.17. Let $X := X_1 \cup_N X_2$ with $\partial X_1 = Y_1^- \sqcup N$ and $\partial X_2 = N^- \sqcup Y_2$. Then, for $n = \dim X$:

i) $\log T^{\mathrm{res},\omega}_{X,R}(\rho) = \frac{n}{2}\chi(X, \partial X)$ and $\log T^{\mathrm{res},\omega}_{X,A}(\rho) = \frac{n}{2}\chi(X)$;
ii) $\log T^{\mathrm{res},\omega}_{X,A}(\rho) - \log T^{\mathrm{res},\omega}_{X,R}(\rho) = \log T^{\mathrm{res},\omega}_{Y}(\rho) + \frac{1}{2}\chi(Y) = \frac{n}{2}\chi(Y)$;
iii) $\log T^{\mathrm{res},\omega}_{X,R}(\rho) = \log T^{\mathrm{res},\omega}_{X_1,R}(\rho) + \log T^{\mathrm{res},\omega}_{X_2,R}(\rho) + \log T^{\mathrm{res},\omega}_{Y}(\rho) + \frac{1}{2}\chi(Y)$;
iv) $\log T^{\mathrm{res},\omega}_{X,A}(\rho) = \log T^{\mathrm{res},\omega}_{X_1,A}(\rho) + \log T^{\mathrm{res},\omega}_{X_2,A}(\rho) - \log T^{\mathrm{res},\omega}_{Y}(\rho) - \frac{1}{2}\chi(Y)$.

PROOF. It follows from Proposition 2.22 and 2.23, [**83**], after observing (6.5.6).

□

COROLLARY 6.5.18. For each $n = \dim X$, $\sum_{k=0}^{n}(-1)^k k \zeta_{k,\mathcal{R}}(0)$ is a topological invariant. In paricular, if $n = \dim X$ is even, then the trace logarithm is additive, i.e.

$$\sum_{k=0}^{n}(-1)^k k \log \Delta^{X}_{k,\mathcal{B}} = \sum_{k=0}^{n}(-1)^k k \log \Delta^{X_1}_{k,\mathcal{B}} + \sum_{k=0}^{n}(-1)^k k \log \Delta^{X_2}_{k,\mathcal{B}},$$

(where $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{A}$) in the Boutet de Monvel calculus, modulo smoothing operators.

PROOF. Both statements follows directly from Theorem 6.5.17. In particular, the second follows also because the residue trace is the unique trace in the Boutet de Monvel calculus.

□

REMARK 6.5.19. As for Corollary 6.5.15, we can conjecture that the formula

$$\chi'(X, \partial X) = \chi'(X_1, \partial X_1) + \chi'(X_2, \partial X_2) + \chi'(N) + \frac{1}{2}\chi(N)$$

may hold for the odd-dimensional case, but not for the even dimensional one. In fact, if we use the values of Remark 6.5.16 for the splitting of S^n along S^{n-1}, we can check that the formula holds if and only if n is odd.

Luckily, quasi-additivity holds for analytic torsion (Theorem 1.1, [**83**]):

$$\log T_{X_1 \cup_{Y_1} X_2}(\rho) = \log T_{X_1}(\rho) + \log T_{X_2}(\rho) + \log T_{Y_1}(\rho).$$

Hence for $\dim X_i$ odd we have a proper gluing formula for the logarithm of the analytic torsion:

$$\log T_{X_1 \cup_{Y_1} X_2}(\rho) = \log T_{X_1}(\rho) + \log T_{X_2}(\rho).$$

This could be seen as the character of a LogTQFT, which we were not able to identify at this stage. We will leave this for future work.

Concluding remarks

The categorification provided by log-functors can form a framework for the study of manifold invariants. In fact, one of the goals of this thesis was to show that such categorification can be generalized to fit more complicated structures and delicate situations, such as invariants in the context of noncommutative geometry, in the hope to understand better additive manifold invariants and possibly find new ones by composition with other traces or quasi-traces (like the case of residue analytic torsions).

As for further problems and projects arising from this research, there are several ones that came to our attention and we would like to study for the future. Indeed, there are other interesting extensions that could be investigated, such as a definition of log-functors for (∞, n)-categories, which should lead to a conjectural *logarithmic cobordism hypothesis*, analogous to the Baez-Dolan cobordism hypothesis for TQFTs ([**48**]). On $(\infty, 2)$-categories, such log-functors should provide a functorial setting for invariants of manifolds with corners. Moreover, it should be possible to extend the Unoriented Logarithm Theorem (Corollary 1.4.42) to $\mathbf{Cob}_n$ for generic n. We expect this to be possible by generalizing the proof with handlebody methods for higher dimensional cobordisms.

On another side, the derived Euler characteristic is just one of a whole family of *higher* Euler characteristics ([**63**]). Its presence in the context of residue torsion suggests that there is more to investigate about the relationship between these Euler characteristics and Deitmar's higher analytic torsions ([**20**]). Also, as mentioned at the end of Chapter 6, one can attempt to characterize (relative or absolute) residue and analytic torsions for manifolds with boundary in terms of a LogTQFT. As a matter of fact, they are generalized logarithms.

From the family point of view, we defined a family residue torsion via Paycha and Scott's generalization to families of the residue and classical trace ([**62**]). Therefore, by using the ζ-trace for families, we could define a family analytic torsion as the (quasi-)trace-character of the family torsion logarithm, which we expect to be related to Bismut and Lott's family analytic torsion ([**9**]).

Also, the Whitehead torsion of a manifold can be seen as the trace-character of a LogTQFT on the subcategory of h-cobordisms and corresponds to the Reidemeister torsion of the boundary. Our aim in this area is to show a family version of this result and prove that Igusa-Klein torsions ([**24**]) can be seen as characters of a higher log-functor.

Finally, when we were working with index theory of Dirac operators on Hilbert Modules over C^*-algebras, we remarked that there is not much that we know about the Calderón projector in this setting. Hence, we would like to study the Calderón defined in [**1**] and try to prove in this context the conjecture that the index of the realization of an elliptic pseudodifferential operator with respect to the Calderón projector vanishes.

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